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advanced methods in stochastic dynamics of non-linear systems

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Advanced Methods in Stochastic Dynamics of Non-Linear Systems

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INTRODUCTION

This text is based on the lecture notes for the courses on advanced methods in stochastic dynamics of non-linear systems taught in November 1994, February 1995 and December 1997 for Ph. D. students at the Department of Building Technology and Structural Engineering at Aalborg University and in September 1999 for postgraduate students and academic staff members at the School of Mechanical Engineering of the University of the Witwatersrand, Johannesburg. The book covers, in authors belief, the most important methods for the analysis of non-linear mechanical, or structural, systems. A large part of the methods covered in the book, especially of those relevant to random pulses, have been developed by the authors. The main idea is to present the methods of Markov processes in the widest possible context. Therefore different stochastic processes with independent increments, leading to the Markov processes problems, are introduced. Moreover some non-Markov problems are discussed, which can be exactly converted into Markov problem. A key point for all the techniques devised subsequently is the general integro-differential Chapman-Kolmogorov equation, known in physics as the master equation.

In Chapter 1 three fundametal processes with independent increments are characterized, i.e. the Wiener process, the compound Poisson process and the α-stable Lévy motion. These processes are regarded as the so-called generating sources, which need not be directly the excitation processes, but may be the ultimate input processes, and the actual random excitations may be modelled with the help of them. Also, in this chapter the construction of the state vector of the dynamical system, in the most general case is given and a heuristic demonstration when the state vector can be regarded as a Markov process.

Chapter 2, which is an important prerequisite for the subsequent chapters presents the derivation of the forward and backward integro-differential Chapman-Kolmogorov equation which is the name adopted for the basic equation specifying the development of the joint probability density function due to excitations from continuous and discontinuous processes with independent increments. In this chapter also the derivation of the generalized Itô's differential rule is given.

Chapter 3 covers the diffusive Markov process techniques, suitable for hysteretic and non-hysteretic non-linear systems under Gaussian white noise and filtered Gaussian white noise excitations. It provides a review of available analytical solutions of the Fokker-Planck-Kolmogorov equation and presents the moment equations and closure approximations technique. Asymptotic expansions of the multivariate probability density functions are also discussed.

Basic methods of stochastic point processes are covered in Chapter 4. General point processes as well as the Poisson process and renewal process are characterized. The filtered point processes, useful in modelling the random trains of overlapping pulses, are introduced. The random trains of non-overlapping pulses treated with the help of queueing theory methods are also dealt with.

In Chapter 5, the non-diffusive Markov process techniques, suitable for non-linear dynamical systems under Poisson trains of impulses are presented. It provides a review of the available analytical solutions for the response probability density. The moment equations technique and the cumulant neglect closure technique, modified for the case of Poisson impulses is developed.

Chapter 6 deals with non-Markov reponse problems convertible to Markov problems. First, the non-linear dynamical systems subjected to random trains of impulses driven by Erlang renewal processes are considered. It is shown how the Erlang impulse process is expressed in terms of the Poisson counting process and consequently the non-Markov response is reduced to a Markov problem at the expense of introducing additional, auxiliary state variables. The technique presented includes the derivation of the stochastic equations for these variables. An extension of this approach, which allows to cover a wider class of renewal driving processes is also presented. Another non-Markov problem discussed in this chapter deals with the Poisson train of general pulses, with sine half-wave shapes.

Chapter 7 provides an extensive account of the methods of treatment of first-passage time problems. After a prerequisite statement of problems, the Markov systems and the crossing theory, including the integral equations approach, are presented in detail.

In Chapter 8, the cell-to-cell mapping techniques, also called path integration techniques are covered. The general idea of state space and time discretization is introduced and next different techniques for white noise driven systems are presented. Two versions of the cell-to-cell mapping technique suitable for the systems driven by the Poisson impulse process are devised. It is also shown how these techniques may be extended to the case of the system driven by an Erlang impulse process.

Chapter 9 widely presents the Petrov-Galerkin method of solving the Fokker-Planck equation and backward Kolmogorov equation as well as their integro-differential counterparts for the Poisson pulse problems, i.e. forward and backward Kolmogorov-Feller equations. The version of the Galerkin method presented consists in expanding the unknown probability density function in series of approximating shape functions and expanding the variational field in series of weighting functions. An upwind differencing in the weighting functions is introduced in order to achieve the numerical stability.

The last, Chapter 10, covers the techniques of equivalent systems, such as the equivalent linearization and equivalent polynomial expansion techniques. The versions of the techniques for the systems driven by the Gaussian white noise and by the Poisson impulse process are presented.

We hope that this text will be helpful to those who theoretically study random vibrations of mechanical and structural systems and, more widely, to those who are interested in stochastic modelling of various dynamical phenomena.

We would like to thank Mrs. Solveig Hesselvang for typing the manuscript and Mrs. Norma Hornung for preparing the drawings.

Radosław Iwankiewicz
Søren R. K. Nielsen

CHAPTER 1

GENERATING SOURCES AND MODELLING ASSUMPTIONS

In this section various models for the generating source of a vibratory system are presented. By a generating source we mean the final stochastic excitation on a vibratory system, and not merely the loadings. The actual dynamic loadings may also be obtained by a filtration of the generating sources.

The main distinction will be made between generating source processes, which have sample curves, that are continuous with probability 1, and those, which may be discontinuous (perform jumps). The scalar and m-dimensional vector continuous source processes are designated $\{W(t), t \in [0, \infty[\}$ and $\{\mathbf{W}(t), t \in [0, \infty[\}$, respectively. The scalar and l-dimensional vector discontinuous source processes are designated $\{V(t), t \in [0, \infty[\}$ and $\{\mathbf{V}(t), t \in [0, \infty[\}$, respectively. A basic approach in the following outline is that all generating source processes have independent increments, i.e. for any $0 \leq t_0 < t_1 < \cdots < t_n$ the stochastic variables $\Delta W(t_0) = W(t_1) - W(t_0), \Delta W(t_1) = W(t_2) - W(t_1), \cdots, \Delta W(t_{n-1}) = W(t_n) - W(t_{n-1})$ are mutually stochastically independent and independent of the initial value $W(t_0)$. This assumption restricts $\{W(t), t \in [0, \infty[\}$ to a *Wiener process*, $\{\mathbf{W}(t), t \in [0, \infty[\}$ to a vector (multivariate) Wiener process, $\{V(t), t \in [0, \infty[\}$ to a *compound Poisson process* or an *α-stable Lévy motion process* and $\{\mathbf{V}(t), t \in [0, \infty[\}$ to a vector (multivariate) compound Poisson process or α-stable Lévy motion.

The dynamic loading processes on a single-degree-of-freedom (SDOF) or a multi-degrees-of-freedom (MDOF) system are designated $\{F(t), t \in [0, \infty[\}$ or $\{\mathbf{F}(t), t \in [0, \infty[\}$, respectively. The corresponding displacement processes are designated $\{Y(t), t \in [0, \infty[\}$ and $\{\mathbf{Y}(t), t \in [0, \infty[\}$, respectively. The n-dimensional state vector process, describing the integrated dynamic system made up of displacements and velocities, possible hysteretic components and filter equations for the loading, is designated $\{\mathbf{Z}(t), t \in [0, \infty[\}$. The basic assumption in the present outline is that the state vector process can be modelled as a Markov vector process.

A stochastic vector process $\{\mathbf{Z}(t), t \in [0, \infty[\}$ is a *Markov process*, if for any $t_1 < t_2 < \cdots < t_{n-1} < t_n$ the following relation exists between the conditional probability density functions

$$\begin{aligned} &f_{\{\mathbf{Z}\}}(\mathbf{z}_n, t_n \mid \mathbf{z}_{n-1}, t_{n-1}; \ldots; \mathbf{z}_2, t_2; \mathbf{z}_1, t_1) = \\ &f_{\{\mathbf{Z}\}}(\mathbf{z}_n, t_n \mid \mathbf{z}_{n-1}, t_{n-1}) = q_{\{\mathbf{Z}\}}(\mathbf{z}_n, t_n \mid \mathbf{z}_{n-1}, t_{n-1}) \end{aligned} \tag{1.1}$$

where $\mathbf{z}_i$ signifies the sample (observations) of $\mathbf{Z}(t_i)$. Relation (1.1) signifies that the probability density function of $\mathbf{Z}(t_n)$ on condition of previous observations $\mathbf{z}_{n-1}, \ldots, \mathbf{z}_2, \mathbf{z}_1$ at the times $t_{n-1}, \ldots, t_2, t_1$ is only inflicted by the latest observation at the time t_{n-1}. The quantity $q_{\{\mathbf{Z}\}}(\mathbf{z}_n, t_n \mid \mathbf{z}_{n-1}, t_{n-1})$ defined in (1.1) is called the transitional probability density function of the Markov vector process. Along with the 1st order probability density $f_{\{\mathbf{Z}\}}(\mathbf{z}_1, t_1)$ at the time t_1, this completely determines the joint probability densities of arbitrary order. This is seen from the following derivation, where the Markov property (1.1) is applied

$$f_{\{\mathbf{Z}\}}(\mathbf{z}_n,t_n\ ;\ \mathbf{z}_{n-1},t_{n-1}\ ;\ \mathbf{z}_{n-2},t_{n-2};\dots;\mathbf{z}_1,t_1)=$$

$$f_{\{\mathbf{Z}\}}(\mathbf{z}_n,t_n\mid\mathbf{z}_{n-1},t_{n-1};\dots;\mathbf{z}_1,t_1)f_{\{\mathbf{Z}\}}(\mathbf{z}_{n-1},t_{n-1}\ ;\ \mathbf{z}_{n-2},t_{n-2}\ ;\dots;\mathbf{z}_1,t_1)=$$

$$q_{\{\mathbf{Z}\}}(\mathbf{z}_n,t_n\mid\mathbf{z}_{n-1},t_{n-1})f_{\{\mathbf{Z}\}}(\mathbf{z}_{n-1},t_{n-1}\mid\mathbf{z}_{n-2},t_{n-2};\dots;\mathbf{z}_1,t_1)$$

$$f_{\{\mathbf{Z}\}}(\mathbf{z}_{n-2},t_{n-2};\dots;\mathbf{z}_1,t_1)=\cdots=$$

$$q_{\{\mathbf{Z}\}}(\mathbf{z}_n,t_n\mid\mathbf{z}_{n-1},t_{n-1})q_{\{\mathbf{Z}\}}(\mathbf{z}_{n-1},t_{n-1}\mid\mathbf{z}_{n-2},t_{n-2})\cdots$$

$$q_{\{\mathbf{Z}\}}(\mathbf{z}_2,t_2\mid\mathbf{z}_1,t_1)f_{\{\mathbf{Z}\}}(\mathbf{z}_1,t_1),\quad t_1<t_2<\cdots<t_{n-1}<t_n \tag{1.2}$$

As an example, consider the generating source $\{W(t),t\in[0,\infty[\}$. Since,

$$W(t_n)=\big(W(t_n)-W(t_{n-1})\big)+\big(W(t_{n-1})-W(t_{n-2})\big)+\cdots+$$

$$\big(W(t_2)-W(t_1)\big)+W(t_1)=\Delta W(t_{n-1})+\Delta W(t_{n-2})+\cdots+\Delta W(t_1)+W(t_1) \tag{1.3}$$

The increments $\Delta W(t_1),\dots,\Delta W(t_{n-1})$ have been assumed to be mutually stochastically independent. Further, $W(t_1)$ is assumed to be independent of all increments. The nth order probability density function of the process can then be written

$$f_{\{W\}}(w_n,t_n\ ;\ w_{n-1},t_{n-1}\ ;\ \cdots\ ;w_2,t_2\ ;\ w_1,t_1)=$$

$$f_{W(t_1)}(w_1)\prod_{j=1}^{n-1}f_{\Delta W(t_j)}(w_{j+1}-w_j)\quad,\quad t_1<t_2<\cdots<t_{n-1}<t_n \tag{1.4}$$

where $f_{\Delta W(t_j)}(w)$ is the probability density function of the increment $\Delta W(t_j)$. Hence, the conditional joint probability density function becomes

$$f_{\{W\}}(w_n,t_n\mid w_{n-1},t_{n-1};\ \cdots\ ;w_2,t_2\ ;\ w_1,t_1)=$$

$$\frac{f_{\{W\}}(w_n,t_n\ ;\ w_{n-1},t_{n-1}\ ;\ \cdots\ ;w_2,t_2\ ;\ w_1,t_1)}{f_{\{W\}}(w_{n-1},t_{n-1}\ ;\ \cdots\ ;w_2,t_2\ ;\ w_1,t_1)}=$$

$$\frac{f_{W(t_1)}(w_1)\prod_{i=1}^{n-1}f_{\Delta W(t_j)}(w_{j+1}-w_j)}{f_{W(t_1)}(w_1)\prod_{j=1}^{n-2}f_{\Delta W(t_j)}(w_{j+1}-w_j)}=$$

$$f_{\Delta W(t_{n-1})}(w_n-w_{n-1})=f_{\{W\}}(w_n,t_n\mid w_{n-1},t_{n-1}) \tag{1.5}$$

Hence, it has been proved that any generating source process with independent increments and for which the initial value $W(t_1)$ is stochastically independent of any increment is then a Markov process.

In section 1.1 the properties of Wiener, compound Poisson and α-stable Lévy motion processes are described. In section 1.2 the dynamic modelling of loadings obtained by filtering of the generating sources is described and finally in section 1.3 the modelling of dynamic systems and the final formulation of system with Markov properties are given.

1.1 Generating sources with independent increments

1.1.1 Wiener process

A stochastic process $\{W(t), t \in [0, \infty[\}$ is a Wiener (or Brownian motion) process if:

1) $\Pr\{W(0) = 0\} = 1$

2) The process has zero mean, i.e. the mean value function $\mu_W(t) = E[W(t)] \equiv 0$

3) For arbitrary $0 < t_0 < t_1 < \cdots < t_n$ the increments $\Delta W(t_0) = W(t_1) - W(t_0), \Delta W(t_1) = W(t_2) - W(t_1), \cdots, \Delta W(t_{n-1}) = W(t_n) - W(t_{n-1})$ are independent

4) For arbitrary t and Δt the increment $\Delta W(t) = W(t+\Delta t) - W(t)$ has a Gaussian distribution with the zero mean and with the variance

$$E\left[(\Delta W(t))^2\right] = D\Delta t \tag{1.6}$$

where D is a positive constant called the diffusion coefficient. In the following it is assumed, for simplicity, $D = 1$ (a so-called unit intensity Wiener process).

The auto-covariance function of a Wiener process is (assuming $t_2 > t_1$)

$$\kappa_{WW}(t_1, t_2) = E[W(t_1)W(t_2)] =$$

$$E\Big[W(t_1)(W(t_2) - W(t_1)) + W^2(t_1)\Big] =$$

$$E[W^2(t_1)] + E\Big[W(t_1)(W(t_2) - W(t_1))\Big] \tag{1.7}$$

Due to the independence of increments, the second expectation splits, thus

$$\kappa_{WW}(t_1, t_2) = E[W^2(t_1)] + E[W(t_1)]E[W(t_2) - W(t_1)] =$$

$$E\left[(W(t_1) - W(0))^2\right] \tag{1.8}$$

The result, in accordance with 4), is

$$\kappa_{WW}(t_1, t_2) = t_1 \tag{1.9}$$

and taking into account the opposite case, $t_1 > t_2$, one arrives at the result

$$\kappa_{WW}(t_1, t_2) = \min(t_1, t_2) \tag{1.10}$$

Variance of a Wiener process is, of course,

$$\sigma_W^2(t) = t \tag{1.11}$$

Since the increments of the Wiener process are independent and Gaussian distributed, the Wiener process, as the sum of its increments is, of course, a Gaussian process.

Moments of the increment $dW(t) = W(t+dt) - W(t)$ of the Wiener process during the time interval $[t, t+dt[$ (incremental properties) are

$$\begin{aligned}
&E[dW(t)] = 0 \\
&E\left[(dW(t))^2\right] = dt \\
&E[dW(t_1)dW(t_2)] = 0 \quad \text{for} \quad t_1 \neq t_2 \\
&E\left[(dW(t))^{2k+1}\right] = 0 \\
&E\left[(dW(t))^{2k}\right] = 1 \cdot 3 \cdot 5 \cdots (2k-1)(dt)^k
\end{aligned} \tag{1.12}$$

where $k = 1, 2, 3, \ldots$. The last two relationships are just the Gaussian distribution properties. It is seen that the second order moment of the increment is of order dt and the higher order moments are of higher orders in dt.

Since the auto-covariance function $\kappa_{WW}(t_1, t_2)$ is continuous at the diagonal $t_1 = t_2 = t$, the Wiener process is continuous in the mean square, [1.2]. Further, since the 2nd order mixed derivative $\frac{\partial^2}{\partial t_1 \partial t_2}\kappa_{XX}(t_1, t_2)$ does not exist at the diagonal $t_1 = t_2 = t$, the Wiener process is not differentiable in the mean square. The mentioned properties of continuity and differentiability can even be proved to hold with probability 1. In figure 1.1 a sample path of a Wiener process is shown. The ripples on the top of the sample path are assumed to have zero wave height and zero wave length. Due to the ripples the sample path is continuous but not differentiable. It can be shown that the properties of having independent increments and being continuous, uniquely define a Wiener process [1.2].

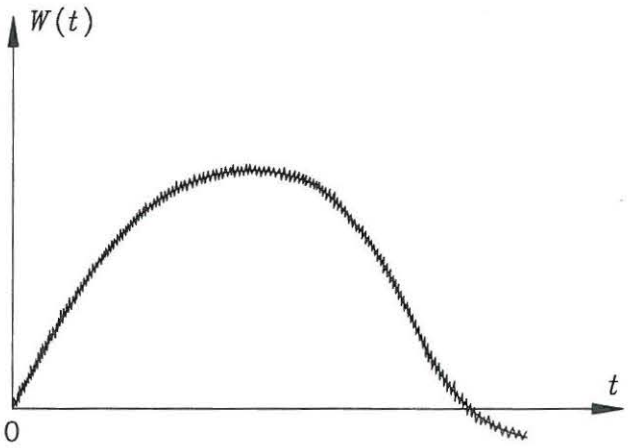

Fig. 1.1. Sample path of a Wiener process.

An m-dimensional Wiener vector process $\{\mathbf{W}(t), t \in [0, \infty[\}$ is defined as a vector process, where all component processes $\{W_\alpha(t), t \in [0, \infty[\}$, $\alpha = 1, \ldots, m$ are assumed to be mutually independent and unit intensity Wiener processes. One then has the cross-covariance function of the process, cf. (1.10)

$$\kappa_{W_\alpha W_\beta}(t_1, t_2) = E\big[W_\alpha(t_1) W_\beta(t_2)\big] = \delta_{\alpha\beta} \min(t_1, t_2) \tag{1.13}$$

where $\delta_{\alpha\beta}$ signifies the Kronecker delta.

1.1.2 Compound Poisson process

A compound Poisson process $\{V(t), t \in [0, \infty[\}$ is represented as

$$V(t) = \sum_{i=1}^{N(t)} P_i \tag{1.14}$$

where $N(t)$ is a Poisson counting process giving the random number of time points t_i in the time interval $[0, t[$ (with the additional assumption: $\Pr\{N(0) = 0\} = 1$, and P_i are independent random variables, identically distributed as a random variable P. Each of the variables P_i is assigned to a random point t_i. The variables P_i are also assumed to be statistically independent of the random times t_i, or of the counting process $N(t)$. A sample path of the process is shown in figure 1.2. Since the counting process counts the number of jumps up to, but excluding the one at the time t, the sample paths are continuous to the left.

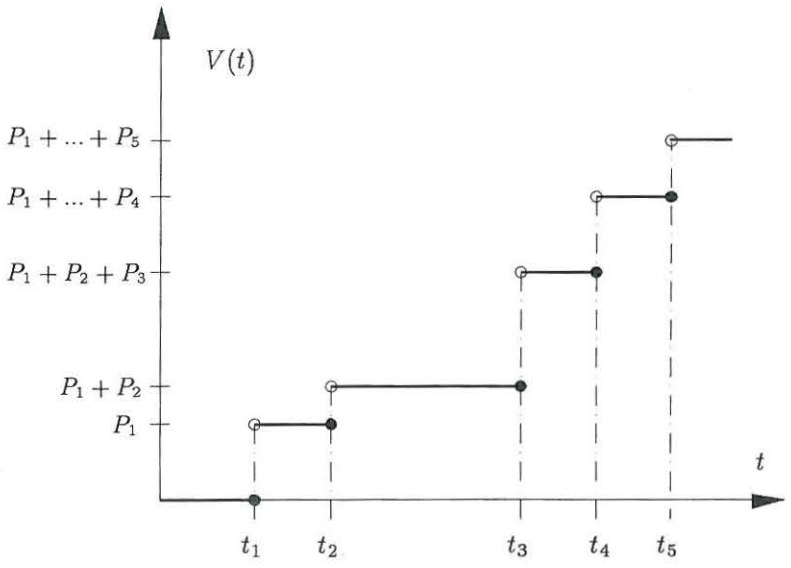

Fig. 1.2. Sample path of a compound Poisson process.

More precisely the compound Poisson process can be presented in the form

$$V(t) = \sum_{i=1}^{N(t)} P_i \hat{\mathbf{1}}(t - t_i) = \sum_{i=1}^{\infty} P_i \hat{\mathbf{1}}(t - t_i) \tag{1.15}$$

where

$$\hat{\mathbf{1}}(t - t_i) = \begin{cases} 1 & , \quad t > t_i \\ 0 & , \quad t \leq t_i \end{cases} \tag{1.16}$$

The indicator function $\hat{\mathbf{1}}(t - t_i)$ is expressed in terms of the usual Heaviside unit step function $\mathbf{1}(t_i - t)$ as

$$\hat{\mathbf{1}}(t - t_i) = 1 - \mathbf{1}(t_i - t) \tag{1.17}$$

seeing that

$$\mathbf{1}(t_i - t) = \begin{cases} 1 & , \quad t_i \geq t \\ 0 & , \quad t_i < t \end{cases} \tag{1.18}$$

Recall that the Poisson process is a regular (orderly) stochastic point process with independent increments. Let the increment of the counting process (the random number of points) in $[t, t + dt[$ be denoted by $dN(t) = N(t + dt) - N(t)$. A counting process is regular if

$$\begin{aligned} &\Pr\{dN(t) = 1\} = \nu(t)dt + \mathrm{O}(dt^2) \\ &\Pr\{dN(t) > 1\} = \mathrm{O}(dt^2) \\ &\Pr\{dN(t) = 0\} = 1 - \nu(t)dt + \mathrm{O}(dt^2) \end{aligned} \tag{1.19}$$

where the order notation $\mathrm{O}(x)$ means that $\mathrm{O}(x) \leq A|x|$, A being a positive constant.

The properties (1.19) mean that the probability of occurrence of one point (event) in the infinitesimal time interval $[t, t + dt[$ is proportional to dt and the probability of occurrence of more than one point is negligibly small.

From (1.19) it follows that

$$E\big[dN(t)\big] = E\Big[\big(dN(t)\big)^n\Big] = \nu(t)dt + \mathrm{O}(dt^2) \tag{1.20}$$

for any n, where $\nu(t)$ is interpreted as a mean arrival (or occurrence) rate of events (points).

Since the Poisson counting process has independent increments

$$E\left[dN(t_1)dN(t_2)\right] = E\left[dN(t_1)\right]E\left[dN(t_2)\right] = \nu(t_1)\nu(t_2)dt_1dt_2, \quad \text{for} \quad t_1 \neq t_2 \quad (1.21)$$

The probability function of the first order of the Poisson counting proccess is given by, [1.5]

$$P_{\{N\}}(n,t) = \Pr\{N(t) = n\} = \frac{\left(\int_0^t \nu(\tau)d\tau\right)^n}{n!} \exp\left(-\int_0^t \nu(\tau)d\tau\right) \quad (1.22)$$

Let us represent the increment $dV(t)$ of the compound Poisson process in the time interval $[t,\ t+dt[$ as

$$dV(t) = P(t)dN(t) \quad (1.23)$$

where $P(t)$ denotes the random variable assigned to the time point occurring in the time interval $[t,\ t+dt[$. Since $P(t_1)$ and $P(t_2)$ for disjoint differential intervals are stochastically independent and independent of the Poisson counting process, and the increments $dN(t_1)$ and $dN(t_2)$ are independent as well, it also follows that the increments $dV(t_1)$ and $dV(t_2)$ are stochastically independent, i.e. the compound Poisson process has independent increments.

By making use of the stochastic independence of $P(t)$ and $dN(t)$ and of the regularity relation (1.19) the following expressions (incremental properties) are obtained

$$\left.\begin{aligned} &E\left[dV(t)\right] = E[P]\nu(t)dt \\ &E\left[\left\{dV(t)\right\}^n\right] = E[P^n]\nu(t)dt \\ &E\left[dV(t_1)dV(t_2)\right] = \left(E[P]\right)^2\nu(t_1)\nu(t_2)dt_1dt_2 \quad \text{for} \quad t_1 \neq t_2 \end{aligned}\right\} \quad (1.24)$$

where it has been assumed that the expectations $E[P^n]$ exist for any n. It is seen in (1.24) that all the moments of the increment $dV(t)$ of the compound Poisson process are non-zero and are of first order in dt.

Integrating (1.23) over time we have

$$V(t) = \int_0^t P(\tau)dN(\tau) \quad (1.25)$$

Making use of (1.20) and of the stochastic independence of $P(\tau)$ and $dN(\tau)$ the following results are obtained for the mean value function $\mu_V(t)$ and the auto-correlation function $\mu_{VV}(t_1,t_2)$

$$\mu_V(t) = E[V(t)] = E\left[\int_0^t P(\tau)dN(\tau)\right] = \int_0^t E[P(\tau)]E[dN(\tau)] = E[P]\int_0^t \nu(\tau)d\tau \quad (1.26)$$

$$\mu_{VV}(t_1,t_2) = E[V(t_1)V(t_2)] = E\left[\int_0^{t_1}\int_0^{t_2} P(\tau_1)P(\tau_2)dN(\tau_1)dN(\tau_2)\right] \tag{1.27}$$

The stochastic integral inside the expectation on the right-hand side of (1.27) is considered as the limit with probability 1 of a Stieltjes double sum of stochastic variables. This sum is divided into the contribution from the diagonal $\tau_1 = \tau_2 = \tau$, which has the limit $\int_0^{\min(t_1,t_2)} P^2(\tau)\big(dN(\tau)\big)^2$, and the off-diagonal terms, where $\tau_1 \neq \tau_2$. For the off-diagonal terms all four stochastic variables $P(\tau_1), P(\tau_2), dN(\tau_1), dN(\tau_2)$ are mutually stochastically independent. Use of the regularity condition (1.20) then provides

$$\mu_{VV}(t_1,t_2) = \int_0^{\min(t_1,t_2)} E\Big[P^2(\tau)\big(dN(\tau)\big)^2\Big] + \int_0^{t_1}\int_0^{t_2} E\Big[P(\tau_1)P(\tau_2)dN(\tau_1)dN(\tau_2)\Big|_{\tau_1\neq\tau_2}\Big]$$

$$= E[P^2]\int_0^{\min(t_1,t_2)} \nu(\tau)d\tau + \big(E[P]\big)^2 \int_0^{t_1}\int_0^{t_2} \nu(\tau_1)\nu(\tau_2)d\tau_1 d\tau_2 \tag{1.28}$$

From (1.26) and (1.28) the auto-covariance function $\kappa_{VV}(t_1,t_2)$ and the variance function are obtained as follows

$$\kappa_{VV}(t_1,t_2) = E[P^2]\int_0^{\min(t_1,t_2)} \nu(\tau)d\tau \tag{1.29}$$

$$\sigma_V^2(t) = E[P^2]\int_0^{t} \nu(\tau)d\tau \tag{1.30}$$

The following derivation is due to Roberts [1.13].

The joint characteristic function of $\mathbf{V}^T = [V(t_1), V(t_2), \ldots, V(t_n)]$ is, by definition, expressed as

$$\Phi_{\mathbf{V}}(\boldsymbol{\theta}, t_1, t_2, \ldots t_n) = E\left[\exp\left(i\sum_{j=1}^{n}\theta_j V(t_j)\right)\right] \tag{1.31}$$

Let us use the following integral representation, corresponding to (1.15)

$$V(t_j) = \int_0^{t_j} P(\tau)dN(\tau) = \int_0^{t} \hat{\mathbf{1}}(t_j - \tau)P(\tau)dN(\tau) \tag{1.32}$$

where t may be any $t > \max(t_j)$. Hence

$$\Phi_{\mathbf{V}}(\boldsymbol{\theta}, t_1, t_2, \ldots t_n) = E\left[\exp\left(i\sum_{j=1}^{n}\theta_j \int_0^t \hat{\mathbf{1}}(t_j - \tau)P(\tau)dN(\tau)\right)\right] \tag{1.33}$$

Let us divide the interval $[0, t[$ into m contiguous subintervals, of length $\Delta\tau_k$ each, i.e. $[\tau_k, \tau_k + \Delta\tau_k[$. The stochastic integral in (1.33) is a limit, in the mean-square sense, of the sequence of the sums

$$\sum_{k=1}^{m} \hat{\mathbf{1}}(t_j - \tau_k)P(\tau_k)\Delta N(\tau_k) \tag{1.34}$$

as $m \to \infty$ and $\Delta\tau_k \to 0$, where $\Delta N(\tau_k) = N(\tau_k + \Delta\tau_k) - N(\tau_k)$ is the increment of the counting process $N(t)$. The expression for the characteristic function may then be represented as

$$\Phi_{\mathbf{V}}(\boldsymbol{\theta}, t_1, t_2, \ldots t_n) = \lim_{\Delta\tau_{\max}\to 0} E\left[\exp\left(i\sum_{j=1}^{n}\theta_j \sum_{k=1}^{m} \hat{\mathbf{1}}(t_j - \tau_k)P(\tau_k)\Delta N(\tau_k)\right)\right] \tag{1.35}$$

where $\Delta\tau_{\max} = \max(\Delta\tau_k)$. Next the order of summation and factorization of the exponential are interchanged. Thus

$$\Phi_{\mathbf{V}}(\boldsymbol{\theta}, t_1, t_2, \ldots t_n) = \lim_{\Delta\tau_{\max}\to 0} E\left[\prod_{k=1}^{m} \exp\left(i\sum_{j=1}^{n}\theta_j \hat{\mathbf{1}}(t_j - \tau_k)P(\tau_k)\Delta N(\tau_k)\right)\right] \tag{1.36}$$

Due to the independence of increments of the counting process and the independence of the random mark variables, the expectation of the product in (1.36) can be split, which gives

$$\Phi_{\mathbf{V}}(\boldsymbol{\theta}, t_1, t_2, \ldots t_n) = \lim_{\Delta\tau_{\max}\to 0} \prod_{k=1}^{m} E\left[\exp\left(i\sum_{j=1}^{n}\theta_j \hat{\mathbf{1}}(t_j - \tau_k)P(\tau_k)\Delta N(\tau_k)\right)\right] \tag{1.37}$$

Each expectation in (1.37) may be performed as

$$E\left[\exp\left(i\sum_{j=1}^{n}\theta_j\hat{\mathbf{1}}(t_j-\tau_k)P(\tau_k)\Delta N(\tau_k)\right)\right]=$$

$$E\left[\exp\left(i\sum_{j=1}^{n}\theta_j\hat{\mathbf{1}}(t_j-\tau_k)P(\tau_k)\cdot 0\right)\middle|\,\Delta N(\tau_k)=0\right]\Pr\{\Delta N(\tau_k)=0\}+$$

$$E\left[\exp\left(i\sum_{j=1}^{n}\theta_j\hat{\mathbf{1}}(t_j-\tau_k)P(\tau_k)\cdot 1\right)\middle|\,\Delta N(\tau_k)=1\right]\Pr\{\Delta N(\tau_k)=1\}+\mathrm{O}(\Delta\tau_k^2)=$$

$$1\cdot\Big(1-\nu(\tau_k)\Delta\tau_k\Big)+\Phi_P\left(\sum_{j=1}^{n}\theta_j\hat{\mathbf{1}}(t_j-\tau_k)\right)\nu(\tau_k)\Delta\tau_k+\mathrm{O}(\Delta\tau_k^2) \tag{1.38}$$

where the stochastic independence of $P(\tau_k)$ and $\Delta N(\tau_k)$ has been used, and the fact that

$$\Phi_P\left(\sum_{j=1}^{n}\theta_j\hat{\mathbf{1}}(t_j-\tau_k)\right)=E\left[\exp\left(iP(\tau_k)\sum_{j=1}^{n}\theta_j\hat{\mathbf{1}}(t_j-\tau_k)\right)\right]=$$

$$E\left[\exp\left(iP\sum_{j=1}^{n}\theta_j\hat{\mathbf{1}}(t_j-\tau_k)\right)\right] \tag{1.39}$$

is, by definition, the characteristic function of the random variable P, the argument being $\sum_{j=1}^{n}\theta_j\hat{\mathbf{1}}(t_j-\tau_k)$ and is irrespective of time, since the mark variables $P(\tau_k)$ are identically distributed.

Hence one has

$$\Phi_{\mathbf{V}}(\boldsymbol{\theta},t_1,t_2,\ldots t_n)=\lim_{\Delta\tau_{\max}\to 0}\prod_{k=1}^{m}\left(1+\left[\Phi_P\left(\sum_{j=1}^{n}\theta_j\hat{\mathbf{1}}(t_j-\tau_k)\right)-1\right]\nu(\tau_k)\Delta\tau_k\right) \tag{1.40}$$

Taking the logarithms of both sides of (1.40) yields

$$\ln\big(\Phi_{\mathbf{V}}(\boldsymbol{\theta},t_1,t_2,\ldots t_n)\big)=$$

$$\lim_{\Delta\tau_{\max}\to 0}\sum_{k=1}^{m}\ln\left(1+\left[\Phi_P\left(\sum_{j=1}^{n}\theta_j\hat{\mathbf{1}}(t_j-\tau_k)\right)-1\right]\nu(\tau_k)\Delta\tau_k\right)=$$

$$\lim_{\Delta\tau_{\max}\to 0}\sum_{k=1}^{m}\left(\left[\Phi_P\left(\sum_{j=1}^{n}\theta_j\hat{\mathbf{1}}(t_j-\tau_k)\right)-1\right]\nu(\tau_k)\Delta\tau_k+\mathrm{O}(\Delta\tau_k^2)\right)=$$

$$\int_0^t\left[\Phi_P\left(\sum_{j=1}^{n}\theta_j\hat{\mathbf{1}}(t_j-\tau)\right)-1\right]\nu(\tau)d\tau \tag{1.41}$$

where the MacLaurin expansion $\ln(1+x)=x+\mathrm{O}(x^2)$ has been used.

The log-characteristic function (1.41) can be expanded in MacLaurin series in terms of θ as follows

$$\ln\bigl(\Phi_{\mathbf{V}}(\boldsymbol{\theta},t_1,t_2,\dots t_n)\bigr)=$$

$$\sum_{j=1}^{n}\lambda_1\bigl[V(t_j)\bigr](i\theta_j)+\frac{1}{2!}\sum_{j,k=1}^{n}\lambda_2\bigl[V(t_j)V(t_j)\bigr](i\theta_j)(i\theta_k)+\cdots \tag{1.42}$$

where

$$\lambda_n\bigl[V(t_1),\dots,V(t_n)\bigr]=$$

$$E[P^n]\int_0^t\prod_{j=1}^{n}\hat{\mathbf{1}}(t_j-\tau)\nu(\tau)d\tau=E[P^n]\int_0^{\min(t_1,\dots,t_n)}\nu(\tau)d\tau \tag{1.43}$$

and $\lambda_n\bigl[V(t_1),\dots,V(t_n)\bigr]$ is the joint nth order cumulant of the variable $\mathbf{V}^T=\bigl[V(t_1),V(t_2),\dots,V(t_n)\bigr]$.

Since the auto-correlation function (1.28) is continuous at the diagonal $t_1=t_2=t$ it follows that the compound Poisson process is continuous in mean square. As seen from the typical sample path in figure 1.2, this is certainly not the case with probability 1.

Alternatively, the compound Poisson process may be described by the random measure $M(dt,t,dp,p)$, which gives the random number of jumps during the time interval $[t,t+dt[$ into the differential interval $[p,p+dp[$ of the mark variable P. Since the underlying counting process is regular (i.e. the probability of occurrence of more than one jump in the infinitesimal time interval is negligible) this measure has the following properties

$$\int_{\mathcal{P}}M(dt,t,dp,p)=dN(t) \tag{1.44}$$

$$\Pr\{M(dt,t,dp,p) = M^n(dt,t,dp,p)\} = 1\ ,\ n = 2,3,\ldots \tag{1.45}$$

$$\Pr\{M(dt,t,dp,p) = 1\} = \nu(t) f_P(p) dt dp \tag{1.46}$$

$$\Pr\{M(dt,t,dp,p) = 0\} = 1 - \nu(t) f_P(p) dt dp \tag{1.47}$$

$$\Pr\{M(dt,t,dp_1,p_1) \cdot M(dt,t,dp_2,p_2) = 0\} = 1\ ,\ p_1 \neq p_2 \tag{1.48}$$

$$\Pr\{M(dt_1,t_1,dp_1,p_1) = M(dt_2,t_2,dp_2,p_2) = 1\} =$$

$$\nu(t_1) f_P(p_1) dt_1 dp_1 \nu(t_2) f_P(p_2) dt_2 dp_2\ ,\ t_1 \neq t_2 \tag{1.49}$$

where $\mathcal{P} \subseteq R$ is the sample space of the random variable P, and $f_P(p)$ is its probability density function. Notice that the remainders of the order $O(dt^2)$ and $O(dp^2)$ have not been indicated in (1.45)-(1.49). Relation (1.44) states that the total number $dN(t)$ of jumps in the interval $[t,\ t+dt[$ is obtained by summing up the jumps to all possible intervals $[p,\ p+dp[$. This is so since the possibility of multiple jumps during $[t,\ t+dt[$ has been excluded by the regularity condition. (1.45) is another statement of the regularity condition. Either, $M(dt,t,dp,p) = 0$, or $M(dt,t,dp,p) = 1$. In both cases $M(dt,t,dp,p) = M^n(dt,t,dp,p)$ for any n. Relation (1.46) is a consequence of the mutual independence of the mark variable P and the Poisson counting process. So, the probability of the joint event of a jump during $[t,\ t+dt[$ into the interval $[p,\ p+dp[$ splits into a product of marginal probabilities. Next, (1.48) is another consequence of the regularity condition. For $p_1 \neq p_2$, $M(dt,t,dp_1,p_1)$ and $M(dt,t,dp_2,p_2)$ are both either 0 or 1 but they cannot both be 1, since this would imply two independent jumps to the disjoint intervals $[p_1,\ p_1+dp_1[$ and $[p_2,\ p_2+dp_2[$ during the same time interval $[t,\ t+dt[$. Therefore $M(dt,t,dp_1,p_1) \cdot M(dt,t,dp_2,p_2) = 0$. Finally, (1.49) is a consequence of the independence of increments of the compound Poisson process, so jump events during different time intervals are independent events and the probability of the joint event again splits into a product of marginal probabilities.

From the above properties it follows that

$$E[M(dt,t,dp,p)] = E[M^n(dt,t,dp,p)] = \nu(t) f_P(p) dt dp \tag{1.50}$$

$$E[M(dt_1,t_1,dp_1,p_1) M(dt_2,t_2,dp_2,p_2)] =$$

$$E[M(dt_1,t_1,dp_1,p_1)] E[M(dt_2,t_2,dp_2,p_2)]\quad t_1 \neq t_2 \tag{1.51}$$

If the jump during $[t,t+dt[$ takes place into interval $[p,p+dp[$, i.e. the jump has the magnitude $P(t) = p$ and $M(dt,t,dp,p) = 1$, the increment of the compound Poisson process is $dV(t) = p = pM(dt,t,dp,p)$. Summing up the possible jumps into all the

contiguous intervals, i.e. summing over the whole sample space $\mathcal{P}$ of the random variable P, yields the following expression for the unconditional increment of the compound Poisson process

$$dV(t) = \int_{\mathcal{P}} pM(dt, t, dp, p) \tag{1.52}$$

More generally, it follows that with probability 1

$$\left(dV(t)\right)^n = \left(\int_{\mathcal{P}} pM(dt, t, dp, p)\right)^n = \int_{\mathcal{P}} p^n M(dt, t, dp, p) \tag{1.53}$$

In the context of (1.52) the left-hand side of (1.53) represents an n-fold Stieltjes integral over $\mathcal{P}^n$. However, because of (1.48) all off-diagonal terms cancel, and only the diagonal terms on the right-hand side of (1.53)give the contribution to the integral.

The compound Poisson process (1.14) may be written as the following stochastic integral

$$V(t) = \sum_{i=1}^{N(t)} P_i = \int_0^t \int_{\mathcal{P}} pM(d\tau, \tau, dp, p) \tag{1.54}$$

It may be shown that performing the suitable expectations and evaluating integrals based on (1.50) and (1.51) leads to the same expressions as (1.24), (1.26) and (1.28).

The generalized derivative of the compound Poisson process (see eq. (1.15)) can then formally be represented as the following random train of Dirac delta impulses

$$\frac{d}{dt}V(t) = \sum_{i=1}^{N(t)} P_i \delta(t - t_i) \tag{1.55}$$

where $\delta(x)$ is the Dirac delta function (or, rather pseudofunction), which is the generalized derivative of a unit step function.

Next, a *multivariate compound Poisson process* $\{\mathbf{V}(t),\ t \in [0, \infty[\}$ is defined as an l-dimensional vector process, where all component processes $\{V_\alpha(t),\ t \in [0, \infty[\}, \alpha = 1, \ldots, l$ are mutually statistically independent compound Poisson processes. Each component process is defined by a random measure $M_\alpha(dt, t, dp, p)$ corresponding to a certain mean arrival rate $\nu_\alpha(t)$ and a mark variable P_α.

1.1.3 α–stable Lévy motion

Quite a wide class of random excitation processes with jumps can be described in terms of α–stable Lévy motions, which turn out to have independent increments and discontinuous sample paths. Initially, some fundamental notations and definitions are introduced.

A random variable is said to have a *stable distribution* if it can be represented as the sum of independent, identically distributed random variables whose probability distributions follow the same law as the distribution of this variable. In other words such a variable is called *α–stable.*

As an example consider the random variable

$$Y = X_1 + X_2 \tag{1.56}$$

which is the sum of two independent, identically distributed random variables.

If the variables X_1 and X_2 have both Gaussian distribution $N(\mu_X, \sigma_X^2)$, then by virtue of a theorem stating that any linear transformation of the Gaussian random variable yields another Gaussian variable, Y has the Gaussian distribution with the mean $\mu_Y = 2\mu_X$ and with variance $\sigma_Y^2 = 2\sigma_X^2$. Hence the Gaussian distribution satisfies the stability property of obeying the same probability law as the underlying random variables.

Now let both the variables X_1 and X_2 have negative exponential distribution, governed by the probability density function

$$f_X(x) = \begin{cases} \lambda \exp(-\lambda x) & , \ x \geq 0 \\ 0 & , \ x < 0 \end{cases} \tag{1.57}$$

It may be easily shown that the probability density of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X_1}(y-x) f_{X_2}(x) dx = \begin{cases} \lambda^2 y \exp(-\lambda y) & , \ y \geq 0 \\ 0 & , \ y < 0 \end{cases} \tag{1.58}$$

Hence the variable Y has gamma distribution $\mathrm{Ga}(1, \lambda)$, with parameter $k = 1$ and λ . It may be concluded that the negative exponential distribution is not a stable distribution (has no stability property).

A generalized Central Limit Theorem (see e.g. [1.6]) states that the limit distribution for the sum of independent, identically distributed random variables is a stable distribution. It is a Gaussian distribution (according to the usual Central Limit Theorem) *iff* these variables have finite variance.

There is a large family of distributions known as *α–stable distributions* satisfying the stability condition. The α–stable random variables X, denoted as $X \sim S_\alpha(\sigma, \beta, \mu)$, are defined by the characteristic function expressed in the general form of

$$\Phi_X(\theta) = \begin{cases} \exp\left\{ i\mu\theta - \sigma^\alpha |\theta|^\alpha (1 - i\beta \mathrm{sgn}\,(\theta) \tan \frac{\alpha\pi}{2}) \right\} & , \ \alpha \in]0,1[\text{ or }]1,2] \\ \exp\left\{ i\mu\theta - \sigma|\theta| \right\} & , \ \alpha = 1 \ , \ \beta = 0 \end{cases} \tag{1.59}$$

where $\alpha \in]0, 2]$ is the *index of stability* (the characteristic exponent), $\beta \in [-1, 1]$ is the *skewness (asymmetry) parameter*, $\sigma \in]0, \infty[$ is the *scale (dispersion) parameter* and $\mu \in]-\infty, +\infty[$ is the *shift (location) parameter.*

Unfortunately, the analytical inversion of the characteristic function (1.59) is only feasible in few special cases, e.g. for $S_2(\sigma, 0, \mu)$ which is the Gaussian distribution $N(\mu, \sigma^2)$ with the density function

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \tag{1.60}$$

or in the case $S_1(\sigma, 0, \mu)$ which is the Cauchy distribution with the density function

$$f_X(x) = \frac{1}{\pi} \frac{\sigma}{\sigma^2 + (x-\mu)^2} \tag{1.61}$$

The characteristic property of α–stable random variables is that, except for the case $\alpha = 2$, they have infinite variance and higher order moments, since for $\alpha \in]0, 2[$

$$E[|X|^p] = \infty, \; p \in [\alpha, \infty[$$

$$E[|X|^p] < \infty, \; p \in]0, \alpha[\tag{1.62}$$

For example, if $\alpha = 1$, the distribution has infinite mean value (Cauchy distribution). In the case $\alpha = 2$ (Gaussian distribution) all the moments are, of course, finite.

For different sets of parameters, a wide variety of probability density curves can be obtained (or modelled) with the help of α–stable distributions, both with positive and negative skewness (see figures 1.3 and 1.4). It is seen that positive values of the skewness parameter β correspond to 'positive skewness', where the right-hand side tail of the density curve is heavier, or thicker, than the left hand-side one. A characteristic property of these curves is that they have inverse power (or algebraic) tails, which means that the tails decay more slowly than the tails of the Gaussian distribution (cf. fig. 1.4). Hence the α–stable distributions are suitable in modelling the distributions with 'heavy' tails. This may be relevant to the phenomena in which the jumps, or impulses, occur, since in those cases the high values of the observed quantity are more likely to occur than in the case of a Gaussian process.

An *α–stable Lévy (standard) motion* is a stochastic process $\{V(t), \; t \in [0, \infty[\}$ for which

1) $\Pr\{V(0) = 0\} = 1$

2) $V(t)$ has independent increments

3) For arbitrary t and Δt the increment $\Delta V(t) = V(t + \Delta t) - V(t)$ has an α–stable distribution, $S_\alpha\left((a\Delta t)^{1/\alpha}, \beta, 0\right)$, where a is a positive constant.

An α–stable Lévy motion is a Brownian motion (Wiener process) when $\alpha = 2$ and it is symmetric for $\beta = 0$.

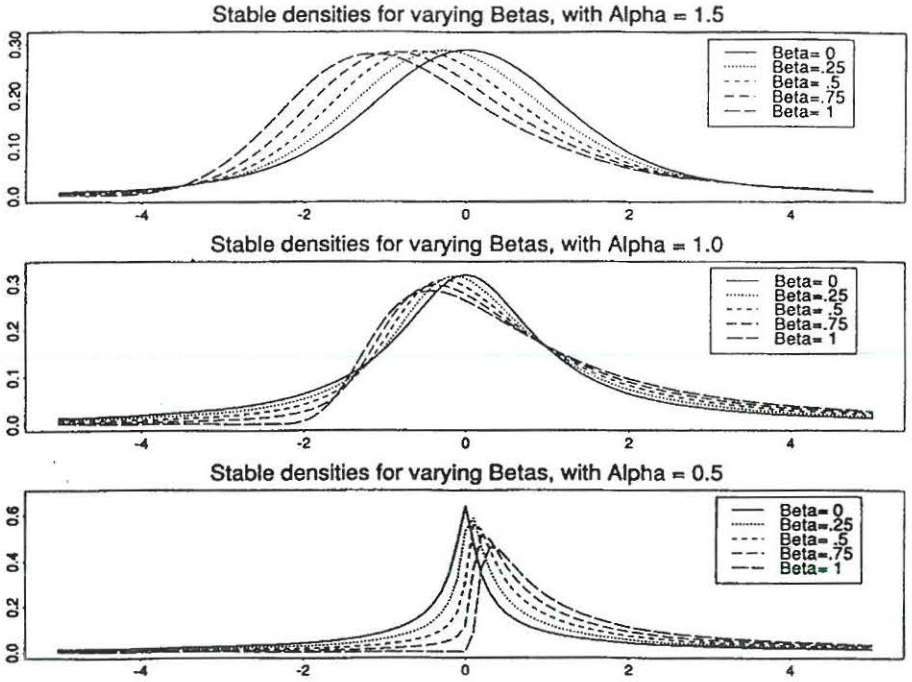

Fig. 1.3. Asymmetric probability density curves of α–stable distributions, [1.9].

The characteristic function of the increment $\Delta V(t)$ during the interval $[t, t+\Delta t[$ becomes

$$\Phi_{\Delta V(t)}(\theta) = \begin{cases} \exp\left\{-(a\Delta t)|\theta|^{\alpha}(1 - i\beta \mathrm{sgn}\,(\theta)\tan\frac{\alpha\pi}{2})\right\} & , \ \alpha \in]0,1[\text{ or }]1,2] \\ \exp\left\{-a\Delta t|\theta|\right\} & , \ \alpha = 1 \ , \ \beta = 0 \end{cases} \tag{1.63}$$

The probability density of an α–stable Lévy motion, which is just the probability density of the jump, may be, in principle, obtained as an inverse Fourier transform of the characteristic function. Thus

$$f_{\Delta V(t)}(p) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp(-i\theta p)\Phi_{\Delta V(t)}(\theta)d\theta \tag{1.64}$$

For example in the case of a Cauchy process one has

$$f_{\Delta V(t)}(p) = \frac{1}{\pi}\frac{a\Delta t}{(a\Delta t)^2 + p^2} \tag{1.65}$$

A random measure $M(dt, t, dp, p)$ may be introduced for the α–stable Lévy motion, defined in the same way as for the compound Poisson process, and with identical properties as follows from (1.44–1.49). Moreover, the representation (1.52) is valid for α–stable Lévy motions as well.

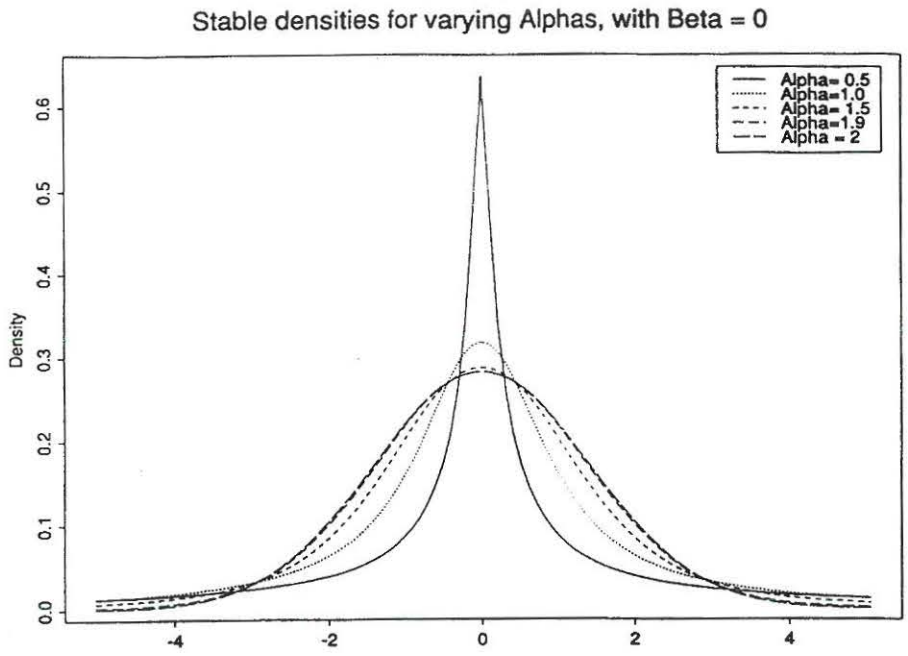

Fig. 1.4. Symmetric probability density curves of α–stable distributions, [1.9].

The probability of making a jump into the interval $[p, p+dp[$ during the time interval $[t, t+\Delta t[$ is given by $f_{\Delta V(t)}(p)dp$. If Δt is sufficiently small this probability can also be expressed as $\Pr\{M(\Delta t, t, dp, p) = 1\}$. Hence one may write

$$\lim_{\Delta t \to 0} \Pr\{M(\Delta t, t, dp, p) = 1\} = \lim_{\Delta t \to 0} f_{\Delta V(t)}(p)dp \tag{1.66}$$

Similarly, a *multivariate α–stable Lévy motion* $\{\mathbf{V}(t),\ t \in [0, \infty[\}$ can be defined as an l-dimensional vector process, where all component processes $\{V_\alpha(t),\ t \in [0, \infty[\}, \alpha = 1, \ldots, l$ are mutually independent and defined by the individual random measures $M_\alpha(dt, t, dp, p)$.

In what follows we shall assume in general that the l-dimensional generating source process $\{\mathbf{V}(t),\ t \in [0, \infty[\}$ is made up of mutually independent component processes $\{V_\alpha(t),\ t \in [0, \infty[\}$, which may be either compound Poisson processes or α–stable Lévy motions. As it will be shown in the subsection 2.1.1 each component process is then completely defined by its so-called *jump probability intensity function* $J_{\{V_\alpha\}}(p_\alpha, t)$ defined as

$$J_{\{V_\alpha\}}(p_\alpha, t) = \lim_{\Delta t \to 0, \Delta p \to 0} \frac{1}{\Delta t \Delta p_\alpha} \Pr\{M_\alpha(\Delta t, t, \Delta p_\alpha, p_\alpha) = 1\} \tag{1.67}$$

Seeing that there may be none jump or one jump in $[t, t+dt[$ (1.67) implies that the expectation $E[M_\alpha(dt, t, dp_\alpha, p_\alpha)]$, if it exists, is evaluated as

$$E[M_\alpha(dt, t, dp_\alpha, p_\alpha)] = J_{\{V_\alpha\}}(p_\alpha, t)dp_\alpha dt \tag{1.68}$$

From (1.46) it follows that the jump probability intensity function of a compound Poisson process is

$$J_{\{V\}}(p,t) = \nu(t) f_P(p) \tag{1.69}$$

For the α–stable Lévy motion from (1.64) and (1.66) it follows that

$$\begin{aligned} J_{\{V\}}(p,t) &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} f_{\Delta V(t)}(p) \\ &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-i\theta p) \Phi_{\Delta V(t)}(\theta) d\theta \end{aligned} \tag{1.70}$$

Upon inserting (1.63) into (1.70) and substituting $x = |p|\theta$ and $u = \frac{a\Delta t}{|p|^\alpha}$ the following result can be obtained

$$J_{\{V\}}(p,t) = \frac{1}{\pi} \frac{a}{|p|^{\alpha+1}} f(\alpha, \beta, \operatorname{sgn}(p)) \tag{1.71}$$

where

$$\begin{aligned} f(\alpha,\beta,\operatorname{sgn}(p)) &= \lim_{u\to 0} \frac{1}{u} \int_0^\infty \cos\left(x\operatorname{sgn}(p) - ux^\alpha \beta \tan\frac{\alpha\pi}{2}\right) \exp(-ux^\alpha) dx \\ &\lim_{u\to 0} \int_0^\infty \left(\beta \tan\frac{\alpha\pi}{2} \operatorname{sgn}(p) \cos\left(x - \operatorname{sgn}(p) u x^\alpha \beta \tan\frac{\alpha\pi}{2}\right)\right. \\ &\left. + \sin\left(x - \operatorname{sgn}(p) u x^\alpha \beta \tan\frac{\alpha\pi}{2}\right)\right) \alpha x^{\alpha-1} \exp(-ux^\alpha) dx \end{aligned} \tag{1.72}$$

The last part follows from the integration by parts of the first part. This last formulation is preferable at numerical applications, since the numerical differentiation is avoided. It should be noted that the integral does not converge uniformly as u approaches zero. Hence, the integration and the limiting operation cannot be interchanged. As seen the function $f(\alpha, \beta, \operatorname{sgn}(p))$ only depends on p through its sign and then it merely acts as different constants for positive and negative values of p. It is then seen that the jump probability intensity function of an α–stable Lévy motion has a singularity of the order $|p|^{\alpha+1}$ as $p \to 0$, i.e. the probability intensity of performing small jumps is much higher than the one of performing larger jumps. For $\beta = 0$ (1.72) provides

$$f(\alpha, 0, \operatorname{sgn}(p)) = \lim_{u\to 0} \int_0^\infty \alpha x^{\alpha-1} \sin(x) \exp(-ux^\alpha) dx \tag{1.73}$$

As it is seen $f(\alpha, 0, \mathrm{sgn}(p))$ is completely independent of p in this case. For $\alpha = 1$ and $\alpha = 2$ (1.72) provides, respectively

$$f(1, 0, \mathrm{sgn}(p)) = 1, \quad f(2, 0, \mathrm{sgn}(p)) = 0 \tag{1.74}$$

Further, it can be shown that $f(\alpha, 0, \mathrm{sgn}(p))$ tends to infinity as $\alpha \to 0$. Relationships (1.71) and (1.74) imply that the jump probability intensity function of the Cauchy process and the Wiener process are, respectively

$$J_{\{V\}}(p, t) = \frac{1}{\pi} \frac{a}{p^2} \tag{1.75}$$

$$J_{\{V\}}(p, t) = 0 \tag{1.76}$$

The result (1.75) can also be proved upon inserting (1.65), (1.66) directly into (1.67).

1.2 Processes obtained by filtering the processes with independent increments

There exist, of course, the excitation processes which have not (not even approximately) independent increments. Consequently, the response to such excitation processes cannot be regarded as Markovian.

One way to remain within the framework of the Markov approach is to regard such excitation processes in a general way as rth order differential form of an auxiliary process, which in turn is the result of filtering the generating source with independent increments through sth order filter ($s > r$). Then the state vector of the system, augmented by the state variables of a filter, as it is governed by the set of first order differential equations driven by the process with independent increments, is a Markov vector process.

Consider the linear SDOF system with mass m, damping ratio ζ and circular eigenfrequency ω_0

$$m(\ddot{Y} + 2\zeta\omega_0\dot{Y} + \omega_0^2 Y) = F(t) \tag{1.77}$$

$F(t)$ is the stochastic excitation process which has not independent increments. Hence the vector process $[Y, \dot{Y}]^T$ is not Markovian.

Let us express the process $F(t)$ as

$$F(t) = p_0 X^{(r)} + p_1 X^{(r-1)} + \cdots + p_r X \tag{1.78}$$

where $X^{(r)}(t) = \frac{d^r}{dt^r} X(t)$ and $X(t)$ as the response of sth order filter, governed by the differential equation

$$X^{(s)} + q_1 X^{(s-1)} + \cdots + q_s X = \dot{V}(t) \tag{1.79}$$

where $p_0, p_1, \ldots, p_r, q_1, \ldots, q_s$ are the real constants. The process $\dot{V}(t)$ stands for the formal derivative of any generating source process with independent increments, i.e. of a Wiener process, of a compound Poisson process, or of an α–stable Lévy motion. Then the augmented state vector $[Y, \dot{Y}, X, \dot{X}, \ldots, X^{(s-1)}]^T$ is a Markov vector process.

The differential filter (1.78), (1.79) can alternatively be given in the following integral form

$$F(t) = \int_0^t h(t-\tau) dV(\tau) \tag{1.80}$$

where

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} H(i\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \frac{P(i\omega)}{Q(i\omega)} d\omega \tag{1.81}$$

and $P(z)$ and $Q(z)$ are polynomials of order r and s, respectively,

$$P(z) = p_0 z^r + p_1 z^{r-1} + \cdots + p_r \tag{1.82}$$

$$Q(z) = z^s + q_1 z^{s-1} + \cdots + q_s \tag{1.83}$$

In (1.80) and (1.81) $h(t)$ and $H(i\omega) = P(i\omega)/Q(i\omega)$ are the *impulse response* and the *frequency response functions* of the shaping filter, respectively. In formulation (1.80) it is assumed that the loading prior to $t = 0$ is zero. If $r < s$ and the roots z_j of $Q(z_j) = 0$ have negative real parts, i.e. $\Re(z_j) < 0$, the integral (1.81) can be evaluated as follows (see e.g. Nielsen [1.11])

$$h(t) = \begin{cases} 0 & , \quad t \leq 0 \\ \sum_{j=1}^{s} e^{z_j t} \frac{P(z_j)}{\prod_{\substack{k=1 \\ k \neq j}}^{s} (z_j - z_k)} & , \quad t > 0 \end{cases} \tag{1.84}$$

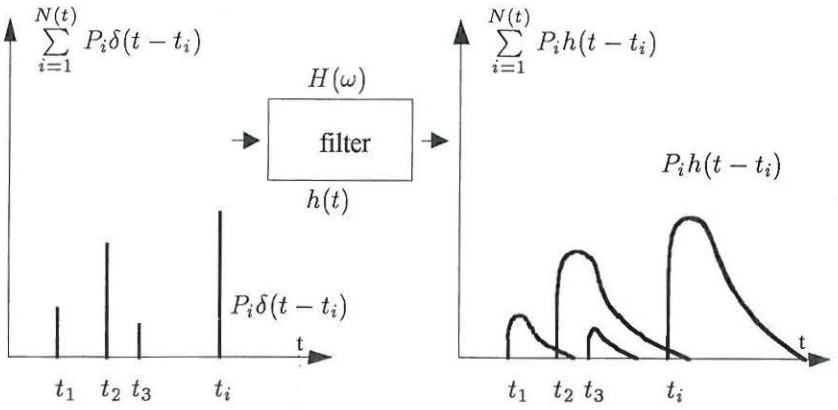

Fig. 1.5 Train of general pulses as a result of filtering of a train of Dirac delta impulses.

The frequency response function of other linear systems can often be approximated by a rational one.

The Poisson-driven train of general pulses may be, in the simplest way, regarded directly as the result of filtering the train of Dirac delta impulses (1.39) through a rational linear filter such as (1.78) and (1.79) (see Fig. 1.5). Upon inserting (1.55) into (1.80) one obtains

$$F(t) = \sum_{i=1}^{N(t)} P_i h(t - t_i) \tag{1.85}$$

where the pulse shape functions $h(t - t_i)$ are just the filter (linear system) impulse response functions.

Thereby the possibilities open of converting many non-Markov response problems to Markov ones.

1.3 Governing stochastic equations in a state vector form

Throughout the outline the non-random dynamical systems, i.e. systems with deterministic coefficients will be considered. The general equation of motion of a non-linear, non-hysteretic SDOF system with the mass m is

$$m\left(\ddot{Y} + g(Y, \dot{Y}) + \frac{\partial}{\partial Y} U(Y)\right) = F(t) \tag{1.86}$$

where $g(Y, \dot{Y})$ is the non-linear and non-conservative part of the restoring force and $\partial U(Y)/\partial Y$ represents the conservative part of the restoring force (U is the potential, or strain energy of the elastic restoring force).

The most important examples of the non-linear, non-hysteretic oscillators are

1) The Duffing oscillator $\left(g(Y, \dot{Y}) = 2\zeta\omega_0\dot{Y},\ U(Y) = \omega_0^2(\frac{1}{2}Y^2 + \frac{1}{4}\varepsilon Y^4)\right)$

$$m\left(\ddot{Y} + 2\zeta\omega_0\dot{Y} + \omega_0^2(1 + \varepsilon Y^2)Y\right) = F(t) \tag{1.87}$$

where the constant ε may assume any values, $\varepsilon \in R$ and ω_0 and ζ are the natural frequency and the damping ratio of the corresponding linear oscillator, respectively.

2) The Rayleigh oscillator $\left(g(Y, \dot{Y}) = -\mu(1 - \alpha\dot{Y}^2)Y,\ U(Y) = \frac{1}{2}\omega_0^2 Y^2\right)$

$$m\left(\ddot{Y} - \mu(1 - \alpha\dot{Y}^2)Y + \omega_0^2 Y\right) = F(t) \tag{1.88}$$

where the constants $\alpha > 0$ and $\mu > 0$.

3) The van der Pol oscillator $\left(g(Y,\dot{Y}) = -\mu(1-\alpha Y^2)\dot{Y},\ U(Y) = \frac{1}{2}\omega_0^2 Y^2\right)$

$$m\left(\ddot{Y} - \mu(1-\alpha Y^2)\dot{Y} + \omega_0^2 Y\right) = F(t) \tag{1.89}$$

where again $\alpha > 0$ and $\mu > 0$

The differential equation (1.86) can be recast into the following set of first order equations

$$\frac{d}{dt}\mathbf{Z}(t) = \mathbf{c}\big(\mathbf{Z}(t)\big) + \mathbf{b}F(t) \tag{1.90}$$

$$\mathbf{Z}(t) = \begin{bmatrix} Y(t) \\ \dot{Y}(t) \end{bmatrix}, \mathbf{c}\big(\mathbf{Z}(t)\big) = \begin{bmatrix} \dot{Y}(t) \\ -\frac{1}{m}g(Y,\dot{Y}) - \frac{1}{m}\frac{\partial}{\partial Y}U(Y) \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \tag{1.91}$$

Equation (1.90) is known as the *state vector formulation*, where $\mathbf{Z}(t)$ is the *state vector* and $\mathbf{c}\big(\mathbf{Z}(t)\big)$ is termed the *drift vector*.

In the case of a hysteretic SDOF oscillator the time-history dependence (hereditary property) of the restoring force acting on the mass can be taken into account by introducing an extra state variable Q. The model is shown in figure 1.6 where the restoring force is made up of its hysteretic component $m\omega_0^2(1-\alpha)Q$, its linear elastic part $m\omega_0^2\alpha Y$, and its linear viscous part $m2\zeta\omega_0\dot{Y}$, in parallel. The parameter α which is termed as the secondary to primary (post- to pre-yielding) stiffness ratio specifies the fraction of the linear elastic part of the restoring force, which is active under plastic loadings, due to the strain hardening or strain softening effects. The equation of motion then becomes

$$m\left(\ddot{Y} + 2\zeta\omega_0\dot{Y} + \omega_0^2\big(\alpha Y + (1-\alpha)Q\big)\right) = F(t) \tag{1.92}$$

The hysteretic state variable Q has been introduced as an extra state variable. In order to close the system, a constitutive relation must be introduced, which relates this quantity to the state variables Y and $\dot{Y}$. This is given in differential form as follows

$$\dot{Q} = \kappa(Y,\dot{Y},Q)\dot{Y} \tag{1.93}$$

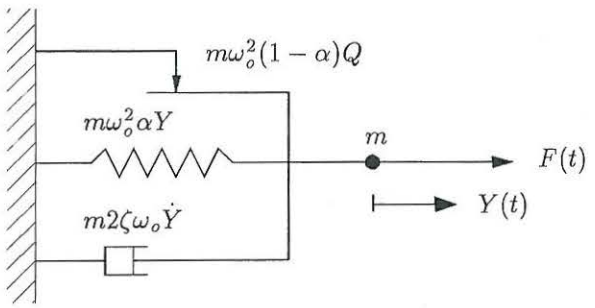

Fig. 1.6. Diagram of hysteretic SDOF system.

where κ may be interpreted as a non-linear state-dependent spring stiffness of the hysteretic component. For a linear elastic material $\kappa \equiv 1$ and (1.93) can be integrated to $Q = Y$. Hence (1.92) reduces to the usual linear oscillator.

The state vector formulation of (1.92), (1.93) is still given by (1.90) with the following new definitions

$$\mathbf{Z}(t) = \begin{bmatrix} Y(t) \\ \dot{Y}(t) \\ Q(t) \end{bmatrix}, \mathbf{c}(\mathbf{Z}(t) = \begin{bmatrix} \dot{Y} \\ -2\zeta\omega_0\dot{Y} - \omega_0^2\big(\alpha Y + (1-\alpha)Q\big) \\ \kappa(Y, \dot{Y}, Q)\dot{Y} \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ \frac{1}{m} \\ 0 \end{bmatrix} \tag{1.94}$$

Various hysteretic models are determined from various constitutive equations. For a bilinear oscillator the non-dimensional spring stiffness assumes the form [1.12]

$$\kappa(\dot{Y}, Q) = 1 - \mathbf{1}(Q - q_0)\big(1 - \mathbf{1}(-\dot{Y})\big) - \mathbf{1}(-Q - q_0)\big(1 - \mathbf{1}(\dot{Y})\big) \tag{1.95}$$

where $\mathbf{1}(x)$ is the Heaviside unit step function defined by (1.16). The quantity q_0 is the yield limit, which is equal to the displacement, at which yielding takes place for the first time. As seen in figure 1.7 a, $\kappa = 0$ when the oscillator is in the elastic range or at the point of moving into this range. The corresponding bilinear behaviour of the total restoring force $\alpha Y + (1-\alpha)Q$ is depicted in figure 1.7 b.

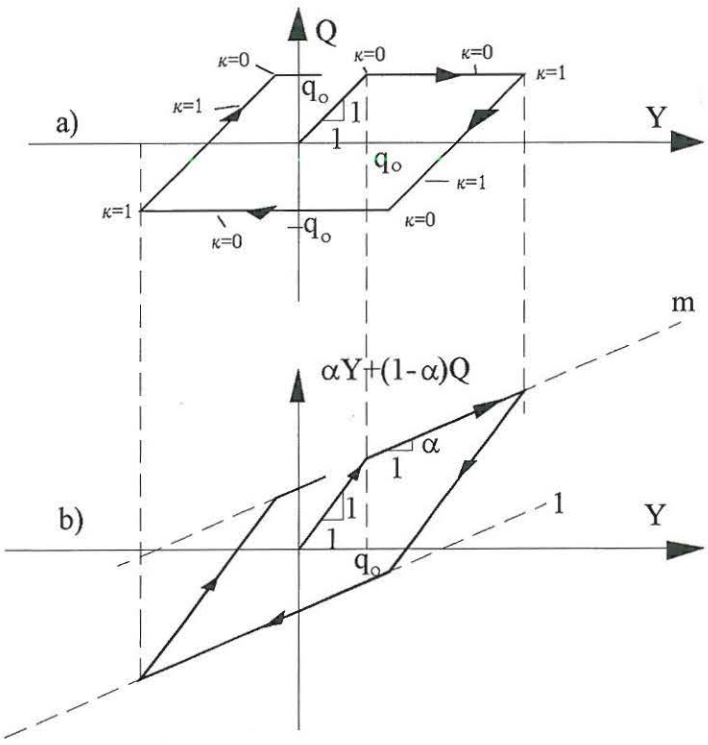

Fig. 1.7. Bilinear oscillator. a) Constitutive relation for the hysteretic state variable, b) Constitutive relation for the total restoring force $\alpha Y + (1-\alpha)Q$.

The Bouc-Wen smooth hysteresis model is given by, [1.14], [1.15]

$$\kappa(\dot{Y}, Q) = 1 - \beta \mathrm{sgn}(\dot{Y}) \frac{Q}{q_0} \left| \frac{Q}{q_0} \right|^{n-1} - \gamma \left| \frac{Q}{q_0} \right|^n \tag{1.96}$$

where β, γ, n are the parameters to be calibrated from available tests. A variety of smoothed hysteretic loops can be modelled by varying β, γ, n. In figure 1.8 the hysteresis loops for $n = 1$ and for different combinations of β and γ are shown. In figure 1.8 a–c the softening and in figure 1.8 d–f the hardening systems are shown, with various degree of hysteresis. Both Q and Y have been non-dimensionalized with respect to the quantity q_0. For $\beta + \gamma = 1$ the quantity q_0 can be identified as the yield displacement, see figure 1.8 a–c. As $n \to \infty$ and $\beta + \gamma = 0.5$ the Bouc-Wen oscillator approaches the bilinear oscillator.

Furthermore, if the loading process $\{F(t),\ t \in [0, \infty[\}$ is the result of the above described filtering, (cf. (1.78), (1.79)) the state vector as given by (1.94) must be augmented by the state variables of the filter $[X, \dot{X}, \ldots, X^{(s-1)}]$ and the augmented state vector is governed by the following system of first order ordinary differential equations obtained from (1.78), (1.79), (1.92), (1.93)

$$\frac{d}{dt}\mathbf{Z}(t) = \mathbf{c}\big(\mathbf{Z}(t)\big) + \mathbf{b}\frac{d}{dt}V(t),\ t > 0 \tag{1.97}$$

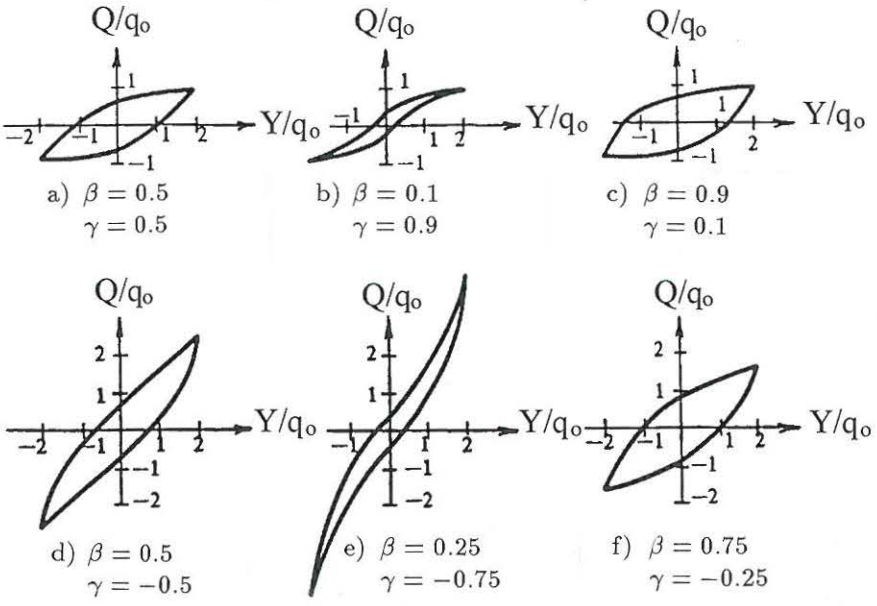

Fig. 1.8. Bouc-Wen hysteretic models, $n = 1$, [1.15].

$$\mathbf{Z}(t) = \begin{bmatrix} Y \\ \dot{Y} \\ Q \\ X \\ \dot{X} \\ \vdots \\ X^{(s-2)} \\ X^{(s-1)} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} \tag{1.98}$$

$$\mathbf{c}(\mathbf{Z}(t) = \begin{bmatrix} \dot{Y} \\ -2\zeta\omega_0\dot{Y} - \omega_0^2(\alpha Y + (1-\alpha)Q) + \frac{p_0}{m}X^{(r)} + \frac{p_1}{m}X^{(r-1)} + \cdots + \frac{p_r}{m}X \\ \kappa(Y, \dot{Y}, Q)\dot{Y} \\ \dot{X} \\ \ddot{X} \\ \vdots \\ X^{(s-1)} \\ -q_1X^{(s-1)} - \cdots - q_sX \end{bmatrix} \tag{1.99}$$

Equation (1.97) may be termed as the state vector formulation in terms of the generating source process, where b_0 is the suitable constant. Equations (1.97) should be solved with suitable initial condition $\mathbf{Z}(0) = \mathbf{Z}_0$, which may be as well deterministic as stochastic.

If a general non-linear hysteretic MDOF system is considered, then with the help of standard techniques of structural dynamics the equations of motion can be written in the matrix form of

$$\mathbf{M}\ddot{\mathbf{Y}} + \mathbf{C}\dot{\mathbf{Y}} + \mathbf{K}_0\mathbf{Y} + \mathbf{K}_1\mathbf{Q} = \mathbf{F}(t) \tag{1.100}$$

where $\mathbf{Y}$ is n_Y-dimensional vector.

In (1.100) the n_Q-dimensional vector $\mathbf{Q}$ signifies the hysteretic state variables. These may be interpreted as the generalized stresses in those structural elements where plasticity is present. As an example in yield-hinge models of elasto-plastic frames, $\mathbf{Q}$ signifies the end-section bending moments of the plastic beam elements, [1.16]. For non-linear elastic ideal plastic structures the constitutive equations can be proved to be, [1.16]

$$\dot{\mathbf{Q}} = \boldsymbol{\kappa}(\dot{\mathbf{Y}}, \mathbf{Q})\dot{\mathbf{Y}} \tag{1.101}$$

Taking into account the possible equations for the hysteretic behaviour and those for a filter, we can build up the state vector of all the generalized coordinates, velocities, hysteretic components and state variables for the filter. The augmented state vector $\mathbf{Z}(t)$ is governed by the equations in the form of (1.97). With due account to the fact that the generating source may consist of an m-variate Wiener processes $\{\mathbf{W}(t),\ t \in [0, \infty[\}$

and an l-variate jump processes $\{\mathbf{V}(t),\ t \in [0,\infty[\}$ the usual differential equations are next converted into the following state vector formulation

$$\begin{aligned} &d\mathbf{Z}(t) = \mathbf{c}\big(\mathbf{Z}(t),t\big)dt + \mathbf{d}d\mathbf{W}(t) + \mathbf{b}d\mathbf{V}(t),\ t>0 \\ &\mathbf{Z}(0) = \mathbf{Z}_0 \end{aligned} \tag{1.102}$$

$$\mathbf{Z}(t) = \begin{bmatrix} \mathbf{Y} \\ \dot{\mathbf{Y}} \\ \mathbf{Q} \\ \mathbf{X} \\ \vdots \\ \mathbf{X}^{(s-1)} \end{bmatrix}, \mathbf{c}(\mathbf{Z}(t) = \begin{bmatrix} \dot{\mathbf{Y}} \\ -\mathbf{M}^{-1}\big(\mathbf{C}\dot{\mathbf{Y}} + \mathbf{K}_0\mathbf{Y} + \mathbf{K}_1\mathbf{Q} + \mathbf{p}_0\mathbf{X}^{(r)} + \cdots + \mathbf{p}_r\mathbf{X}\big) \\ \boldsymbol{\kappa}(\dot{\mathbf{Y}},\mathbf{Q})\dot{\mathbf{Y}} \\ \dot{\mathbf{X}} \\ \vdots \\ -\mathbf{q}_1\mathbf{X}^{(s-1)} - \cdots - \mathbf{q}_s\mathbf{X} \end{bmatrix} \tag{1.103}$$

$$\mathbf{d} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{I} \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{I} \end{bmatrix} \tag{1.104}$$

$\mathbf{p}_0, \cdots, \mathbf{p}_r$ are real matrices of dimension $n_Y \times n_X$ and $\mathbf{q}_1, \cdots, \mathbf{q}_s$ are real matrices of dimension $n_X \times n_Y$, specifying the rational filter for the vector loading $\mathbf{F}(t)$ similar to the scalar filter (1.82), (1.83). In what follows $\mathbf{c}$ will be termed as the *drift vector* and $\mathbf{d}$ the *diffusion matrix.*

Equation (1.102) can be generalized further, assuming that the matrices $\mathbf{d}$ and $\mathbf{b}$ may depend on the state of the system and may depend explicitly on time, so the following formulations are arrived at

$$\left.\begin{aligned} &d\mathbf{Z}(t) = \mathbf{c}\big(\mathbf{Z}(t),t\big)dt + \mathbf{d}\big(\mathbf{Z}(t),t\big)d\mathbf{W}(t) + \mathbf{b}\big(\mathbf{Z}(t),t\big)d\mathbf{V}(t) \\ &\mathbf{Z}(0) = \mathbf{Z}_0 \end{aligned}\right\} \tag{1.105}$$

Throughout this text it is assumed that (1.105) is interpreted in the so-called Itô sense. That means that $d\mathbf{W}(t)$ and $d\mathbf{V}(t)$ indicate the increment of the source processes in the interval $[t, t+dt[$, whereas $\mathbf{d}\big(\mathbf{Z}(t),t\big)$ and $\mathbf{b}\big(\mathbf{Z}(t),t\big)$ carry information of the source processes up to but not including t. Then $\mathbf{d}\big(\mathbf{Z}(t),t\big)$ and $\mathbf{b}\big(\mathbf{Z}(t),t\big)$ become stochastically independent of $d\mathbf{W}(t)$ and $d\mathbf{V}(t)$. Finally the following statement can be given:

If the dynamic behaviour of the system is governed by the set of first order differential equations (1.105) and the generating source processes have independent increments and are statistically independent of random initial conditions $\mathbf{Z}_0$, then the state vector $\{\mathbf{Z}(t), t \in [0,\infty[\}$ is a Markov process.

The following (heuristic) proof has been given by Snyder [1.5] in the case of the compound Poisson driving process but it is certainly valid for any driving processes with independent increments.

The governing stochastic differential equations (1.105) together with the random initial condition can be converted into the integral form of

$$\mathbf{Z}(t) = \mathbf{Z}_0 + \int_0^t \mathbf{c}\big(\mathbf{Z}(\tau),\tau\big)d\tau + \int_0^t \mathbf{d}\big(\mathbf{Z}(\tau),\tau\big)d\mathbf{W}(\tau) + \int_0^t \mathbf{b}\big(\mathbf{Z}(\tau),\tau\big)d\mathbf{V}(\tau) \quad (1.106)$$

Initially the interval $[0,t[$ is split into two subintervals $[0,s[$ and $[s,t[$.

The integral equation (1.106) can then be written as

$$\mathbf{Z}(t) = \mathbf{Z}(s) + \int_s^t \mathbf{c}\big(\mathbf{Z}(\tau),\tau\big)d\tau + \int_s^t \mathbf{d}\big(\mathbf{Z}(\tau),\tau\big)d\mathbf{W}(\tau) + \int_s^t \mathbf{b}\big(\mathbf{Z}(\tau),\tau\big)d\mathbf{V}(\tau) \quad (1.107)$$

and

$$\mathbf{Z}(s) = \mathbf{Z}_0 + \int_0^s \mathbf{c}\big(\mathbf{Z}(\tau),\tau\big)d\tau + \int_0^s \mathbf{d}\big(\mathbf{Z}(\tau),\tau\big)d\mathbf{W}(\tau) + \int_0^s \mathbf{b}\big(\mathbf{Z}(\tau),\tau\big)d\mathbf{V}(\tau) \quad (1.108)$$

Hence $\mathbf{Z}(s)$ as evaluated from (1.108) is an initial condition for the equation (1.107). As it is seen in (1.108) $\mathbf{Z}(s)$ is only a function of $\mathbf{Z}_0$ and of the processes $\{\mathbf{V}(\tau),\tau \in [0,s[\}$ and $\{\mathbf{W}(\tau),\tau \in [0,s[\}$. Since the processes $\{\mathbf{V}(t),t \in [0,\infty[\}$ and $\{\mathbf{W}(t),t \in [0,\infty[\}$ both have independent increments and are statistically independent of the initial conditions $\mathbf{Z}_0$, $\mathbf{Z}(s)$ is statistically independent of $\{\mathbf{V}(\tau),\tau \in [s,\infty[\}$ and $\{\mathbf{W}(\tau),\tau \in [s,\infty[\}$. Hence, $\mathbf{Z}(s)$ is statistically independent of $\int_s^t \mathbf{b}(\mathbf{Z}(\tau),\tau)d\mathbf{V}(\tau)$ and of $\int_s^t \mathbf{d}(\mathbf{Z}(\tau),\tau)d\mathbf{W}(\tau)$.

In turn $\mathbf{Z}(t)$, as it is seen in (1.107), is only determined by the increments of the processes $\{\mathbf{V}(\tau),\tau \in [s,t[\}$ and $\{\mathbf{W}(\tau),\tau \in [s,t[\}$ and by the initial condition $\mathbf{Z}(s)$ which is statistically independent of these processes.

It is concluded that since $\mathbf{Z}(s)$ is determined by $\{\mathbf{Z}(\tau),\tau \in [0,s[\}$ and since the processes $\{\mathbf{V}(\tau),\tau \in [s,t[\}$ and $\{\mathbf{W}(\tau),\tau \in [s,t[\}$ are independent of $\{\mathbf{Z}(\tau),\tau \in [0,s[\}$ (by virtue of independent increments and independence of the initial conditions $\mathbf{Z}_0$): *the conditional distribution of* $\mathbf{Z}(t)$, *given* $\{\mathbf{Z}(\tau),\tau \in [0,s]\}$ *is the same as the conditional distribution* $\mathbf{Z}(t)$ *given* $\mathbf{Z}(s)$ *alone.*

Hence $\mathbf{Z}(t)$ *satisfies the Markov property.*

1.4 Concluding remarks and comments

In this chapter three classes of stochastic processes with independent increments have been introduced and shortly characterized. It has been discussed under which circumstances these processes may be valid models of some actual random excitations and

how some other actual random excitations can be expressed in terms of these processes. Thus the attention has been focused on the processes for which the response of the dynamical system may be regarded as a Markov process. The main reason for such a choice is that for this class of processes there exists a wide variety of analytical methods. Mostly these methods are covered in this book.

For deeper and more detailed studies of the problems outlined in this chapter the Readership is referred to the following books and papers, the list of which is the result of a painful process of selection from the extensive body of existing literature.

1.5 Bibliography

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Let us briefly characterize the contents of these references, under the reservation that the characteristics are only made from the point of view of problems covered in this chapter.

The book [1.1] gives a wide account of methods of analysis for random vibration problems. In particular the problems of Markov processes are covered and the questions connected with treating the actual, physically realizable random excitations, and hence questions of Markov approximation of actual response processes, are explained.

The reference [1.2] is one of the most classical books on the stochastic differential equations. It contains the chapters on the Markov processes, diffusion processes and the Wiener process. It requires from the reader, as Author writes, 'the mathematical preparation usual for the students of physical and engineering sciences'.

The reference [1.3] is one of the most recent books on stochastic differential equations oriented towards the applications. Besides the usual questions of Markov processes, diffusion processes and Wiener process it also covers some other processes with independent increments belonging to the class of Poisson processes. Also the stochastic differential equation for the processes with jumps, driven by the compound Poisson process is discussed.

The books [1.4] and [1.5] are recommended for thorough and detailed studies on the stochastic point processes, in the case of a need of studying the problems of random impulses and processes with jumps. The book [1.5] covers the problems of the so-called Poisson-driven Markov processes, i.e. non-diffusive Markov processes driven by the compound Poisson process, and it also covers the Poisson random measure for this class of processes.

The papers [1.6] and [1.7] may serve as an introduction to 'a wonderful world of α–stable variables and processes' as the Authors of the paper [1.7] have it. For more systematic studies in this field the Reader is referred to the books [1.8] and [1.9]. In the book [1.8], for example, the simulation technique is given for evaluating the probability density of the solution of non-linear stochastic differential equations driven by α–stable processes. It should be emphasized that use of α–stable processes in idealizing different processes revealing 'chaotic' behaviour and jumps has been meeting for some time with an ever increasing interest, see [1.10].

1.6 Example problems

1.1 Making use of integral representation (1.25) and of (1.24) evaluate the mean value, covariance and variance function of the compound Poisson process as given by (1.26), (1.29) and (1.30), respectively. Specify the results for the homogeneous Poisson process.

1.2 Evaluate the same statistics of the compound Poisson process making use of the integral (1.52) with respect to the Poisson random measure and of its properties (1.50) and (1.51).

1.3 By means of the relationships between the characteristic function and the moments show that the properties (1.62) of the α-stable random variables hold.

1.4 Evaluate the spectral density of the response of a linear filter to a white noise input process:

a) in the case of first-order filter,

b) in the case of second-order filter.

Discuss how the results depend on filter parameters.

1.5 What are the shape functions of the pulses obtained by filtering the Dirac delta impulses through:

a) a first-order linear filter?

b) a second-order linear filter?

CHAPTER 2

MASTER EQUATION FOR MARKOV PROCESSES

2.1 Derivation of the forward and backward integro-differential Chapman-Kolmogorov equation

2.1.1 General relationships and sample paths properties

The Markov property is expressed by (1.1) in terms of the transitional probability density $q_{\{\mathbf{Z}\}}(\mathbf{z}_2, t_2|\mathbf{z}_1, t_1)$.

For an arbitrary (not necessarily Markovian) stochastic process for arbitrary $t_3 > t_2 > t_1$ the following relationships hold

$$q_{\{\mathbf{Z}\}}(\mathbf{z}_3, t_3; \mathbf{z}_1, t_1) = \int q_{\{\mathbf{Z}\}}(\mathbf{z}_3, t_3; \mathbf{z}_2, t_2; \mathbf{z}_1, t_1) d\mathbf{z}_2$$

$$= \int q_{\{\mathbf{Z}\}}(\mathbf{z}_3, t_3|\mathbf{z}_2, t_2; \mathbf{z}_1, t_1) q_{\{\mathbf{Z}\}}(\mathbf{z}_2, t_2; \mathbf{z}_1, t_1) d\mathbf{z}_2 \tag{2.1}$$

and consequently

$$q_{\{\mathbf{Z}\}}(\mathbf{z}_3, t_3|\mathbf{z}_1, t_1) = \int q_{\{\mathbf{Z}\}}(\mathbf{z}_3, t_3|\mathbf{z}_2, t_2; \mathbf{z}_1, t_1) q_{\{\mathbf{Z}\}}(\mathbf{z}_2, t_2|\mathbf{z}_1, t_1) d\mathbf{z}_2 \tag{2.2}$$

where the integration is performed over the entire sample space of the random vector $\mathbf{Z}(t_2)$. Using in (2.2) the Markov property (1.1) the following consistency condition for the conditional probability densities is obtained which must hold for any Markov process

$$q_{\{\mathbf{Z}\}}(\mathbf{z}_3, t_3|\mathbf{z}_1, t_1) = \int q_{\{\mathbf{Z}\}}(\mathbf{z}_3, t_3|\mathbf{z}_2, t_2) q_{\{\mathbf{Z}\}}(\mathbf{z}_2, t_2|\mathbf{z}_1, t_1) d\mathbf{z}_2 \tag{2.3}$$

Equation (2.3) is called the Chapman-Kolmogorov (or Chapman-Kolmogorov-Smoluchowski) equation.

Assuming this integral equation as a startpoint we shall in the following derive its differential counterparts, based on an approach due to Gardiner [2.1]. There are no agreed names to such equations, which we shall designate as the forward and the backward integro-differential Chapman-Kolmogorov equations governing the development of the transition probability density function $q_{\{\mathbf{Z}\}}(\mathbf{z}_2, t_2|\mathbf{z}_1, t_1)$ with respect to the forward time t_2 or the backward time t_1, respectively.

Before doing that, let us state define the condition for the continuity of the sample paths of a Markov process.

The sample paths of a Markov process are, with probability one, the continuous functions of time if for any $\varepsilon > 0$ the following condition is satisfied, [2.1]

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int\limits_{|\mathbf{p}|>\varepsilon} q_{\{\mathbf{Z}\}}(\mathbf{z}+\mathbf{p}, t+\Delta t|\mathbf{z}, t)d\mathbf{p} = 0 \tag{2.4}$$

uniformly in $\mathbf{z}$, t and Δt.

This condition means that if the sample paths are continuous, the probability for the subsequent state $\mathbf{z}+\mathbf{p}$ to be finitely different from the previous state $\mathbf{z}$ must tend to zero faster than Δt. This condition expresses the following property of the continuous sample paths: large increments of the process during a small time interval are improbable, or in other words, during a small, infinitesimal time interval only infinitesimal increments of the process are admitted.

Let us now check the sample paths continuity condition for the Wiener, the compound Poisson and the Cauchy processes with probability 1.

The transition probability density function of the univariate Wiener process is (cf. (1.6))

$$q_{\{W\}}(z+p, t+\Delta t|z, t) = \frac{1}{2\sqrt{\pi D \Delta t}} \exp\left\{-\frac{p^2}{4D\Delta t}\right\} \tag{2.5}$$

where p signifies the magnitude of the increment $\Delta W(t)$ during the time interval $[t, t+dt[$. We examine the limit

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int\limits_{|p|>\varepsilon} \frac{1}{2\sqrt{\pi D \Delta t}} \exp\left\{-\frac{p^2}{4D\Delta t}\right\} dx = \lim_{\Delta t \to 0} \frac{1}{\Delta t} 2\Phi\left(-\frac{\varepsilon}{\sqrt{2D\Delta t}}\right) =$$

$$\lim_{\Delta t \to 0} \frac{\varepsilon}{\sqrt{2D}} \frac{(\Delta t)^{-\frac{3}{2}} \varphi\left(-\frac{\varepsilon}{\sqrt{2D\Delta t}}\right)}{1} = 0 \quad \text{for all } \varepsilon > 0 \tag{2.6}$$

where the L'Hospital rule has been used. Hence the condition of the sample paths continuity is satisfied in the case of a Wiener process.

In the case of the compound Poisson process or an α-stable Lévy motion at the time of occurrence of an impulse the state variables of the system increase by a jump, the height of which is equal to the impulse magnitude. Consequently, the transition probability density function $q_{\{V\}}(z+p, t+dt|z, t)$, which is interpreted as the probability density that the system makes a jump of magnitude p from the state z at the time t during the time interval $[t, t+dt[$ is just, cf. (1.67)

$$q_{\{V\}}(z+p, t+dt|z, t) = \lim_{\Delta p \to 0} \frac{1}{\Delta p} \Pr\{M(dt, t, \Delta p, p) = 1\} = J_{\{V\}}(p, t)dt \tag{2.7}$$

where $M(dt, t, dp, p)$ is the random measure of the jump process and $J_{\{V\}}(p, t)$ is the jump probability intensity function.

The condition for the sample paths continuity becomes, cf. (1.67)

$$\int_{|p|>\varepsilon} J_{\{V\}}(p,t)dp = 0 \tag{2.8}$$

Upon inserting (1.69) and (1.71) into (2.8) it is seen that neither the compound Poisson process nor the α-stable Lévy motion has continuous sample curves with probability 1. Continuity of the sample paths of Wiener process may be formulated in such a way that its jump probability intensity function vanishes, $J_{\{W\}}(p,t) \equiv 0$, cf. (1.76).

If the sample paths are discontinuous, the following relationship must hold, for all $\varepsilon > 0$, uniformly in $\mathbf{p}+\mathbf{z}, \mathbf{z}$, and t and for $|\mathbf{p}| > \varepsilon$

$$\int_{|\mathbf{p}|>\varepsilon} J_{\{\mathbf{Z}\}}(\mathbf{z}+\mathbf{p}|\mathbf{z},t)d\mathbf{p} > 0 \tag{2.9}$$

where

$$J_{\{\mathbf{Z}\}}(\mathbf{x}|\mathbf{z},t) = \lim_{\Delta t\to 0}\frac{1}{\Delta t} q_{\{\mathbf{Z}\}}(\mathbf{x},t+\Delta t|\mathbf{z},t) \tag{2.10}$$

$J_{\{\mathbf{Z}\}}(\mathbf{x}|\mathbf{z},t)$ is denoted the jump probability intensity function on the state vector $\mathbf{Z}(t)$.

The so-called derivate moments are defined as the following limits valid for any $\varepsilon > 0$, and uniformly in $\mathbf{x}, \varepsilon$ and t:

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{|\mathbf{x}-\mathbf{z}|<\varepsilon}(x_i - z_i)q_{\{\mathbf{Z}\}}(\mathbf{x},t+\Delta t|\mathbf{z},t)d\mathbf{x} = C_i(\mathbf{z},t) \tag{2.11}$$

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{|\mathbf{x}-\mathbf{z}|<\varepsilon}(x_i - z_i)(x_j - z_j)q_{\{\mathbf{Z}\}}(\mathbf{x},t+\Delta t|\mathbf{z},t)d\mathbf{x} = D_{ij}(\mathbf{z},t) \tag{2.12}$$

The derivate moments characterize the infinitesimal mean and the variance which exist even for continuous sample path (Wiener process excitations). For a continuous process transitioned probability mass will be confined within the sphere $|\mathbf{x}-\mathbf{z}| < \varepsilon$ as $\Delta t \to 0$ for all ε. Hence as $\Delta t \to 0$ the derivative moments may be interpreted as moments conditioned on the state $\mathbf{z}(t) = 8$ at the time t, i.e.

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}E[Z_i(t+\Delta t) - Z_i(t)|\mathbf{Z}(t) = \mathbf{z}] = C_i(\mathbf{z},t) \tag{2.13}$$

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}E\Big[\big(Z_i(t+\Delta t) - Z_i(t)\big)\big(Z_j(t+\Delta t) - Z_j(t)\big)|\mathbf{Z}(t) = \mathbf{z}\Big] = D_{ij}(\mathbf{z},t) \tag{2.14}$$

Derivative moments above 2nd order are all equal to zero for a continuous Markov process.

2.1.2 Forward integro-differential Chapman-Kolmogorov equation

Let us take an arbitrary, twice continuously differentiable function $f(\mathbf{z})$ and let us examine the evolution in the forward time of the conditional expectation of this function as a function of state variables, hence the time derivative

$$\frac{\partial}{\partial t}\int f(\mathbf{x})q_{\{\mathbf{Z}\}}(\mathbf{x},t|\mathbf{y},t_0)d\mathbf{x} =$$

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int f(\mathbf{x})[q_{\{\mathbf{Z}\}}(\mathbf{x},t+\Delta t|\mathbf{y},t_0) - q_{\{\mathbf{Z}\}}(\mathbf{x},t|\mathbf{y},t_0)]d\mathbf{x} \tag{2.15}$$

where $\mathbf{y}$ siginfies the initial (backward) state and t_0 is the initial (backward) time. Making use of Chapman-Kolmogorov equation (2.3) we obtain for any $t > t_0$

$$\frac{\partial}{\partial t}\int f(\mathbf{x})q_{\{\mathbf{Z}\}}(\mathbf{x},t|\mathbf{y},t_0)d\mathbf{x} =$$

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\{\int\int f(\mathbf{x})q_{\{\mathbf{Z}\}}(\mathbf{x},t+\Delta t|\mathbf{z},t)q_{\{\mathbf{Z}\}}(\mathbf{z},t|\mathbf{y},t_0)d\mathbf{x}d\mathbf{z} -\right.$$

$$\left.\int f(\mathbf{z})q_{\{\mathbf{Z}\}}(\mathbf{z},t|\mathbf{y},t_0)d\mathbf{z}\right\} \tag{2.16}$$

In the first term the order of integration over $\mathbf{z}$ and $\mathbf{x}$ has been changed, and the integration variable has been changed from $\mathbf{x}$ to $\mathbf{z}$ in the last term. Let us divide the domain of integration over $\mathbf{x}$ into two subdomains: $|\mathbf{x}-\mathbf{z}| < \varepsilon$ and $|\mathbf{x}-\mathbf{z}| \geq \varepsilon$, thus

$$\frac{\partial}{\partial t}\int f(\mathbf{x})q_{\{\mathbf{Z}\}}(\mathbf{x},t|\mathbf{y},t_0)d\mathbf{x} =$$

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\{\int\int_{|\mathbf{x}-\mathbf{z}|<\varepsilon} f(\mathbf{x})q_{\{\mathbf{Z}\}}(\mathbf{x},t+\Delta t|\mathbf{z},t)q_{\{\mathbf{Z}\}}(\mathbf{z},t|\mathbf{y},t_0)d\mathbf{x}d\mathbf{z} +\right.$$

$$\int\int_{|\mathbf{x}-\mathbf{z}|\geq\varepsilon} f(\mathbf{x})q_{\{\mathbf{Z}\}}(\mathbf{x},t+\Delta t|\mathbf{z},t)q_{\{\mathbf{Z}\}}(\mathbf{z},t|\mathbf{y},t_0)d\mathbf{x}d\mathbf{z} -$$

$$\left.\int f(\mathbf{z})q_{\{\mathbf{Z}\}}(\mathbf{z},t|\mathbf{y},t_0)d\mathbf{z}\right\} \tag{2.17}$$

When $\Delta t \to 0$ the integrates over $|\mathbf{x}-\mathbf{z}| < \varepsilon$ covers the continuous part and the integral over $|\mathbf{x}-\mathbf{z}| > \varepsilon$ the discontinuous part of the Markov process.

Since $f(\mathbf{z})$ is twice continuously differentiable, for $|\mathbf{x}-\mathbf{z}| < \varepsilon$ the following Taylor expansion can be used

$$f(\mathbf{x}) =$$

$$f(\mathbf{z}) + \sum_i \frac{\partial f(\mathbf{z})}{\partial z_i}(x_i - z_i) + \sum_{i,j} \frac{1}{2}\frac{\partial^2 f(\mathbf{z})}{\partial z_i \partial z_j}(x_i - z_i)(x_j - z_j) + \mathrm{O}\Big(|\mathbf{x}-\mathbf{z}|^3\Big) \qquad (2.18)$$

Next we substitute the expansion (2.18) into the first integral in (2.17), separating the term involving $f(\mathbf{z})$, and the last integral of (2.17) is multiplied by $q_{\{\mathbf{Z}\}}(\mathbf{x}, t+\Delta t|\mathbf{z}, t)$ followed by an integration over $\mathbf{x}$ (which gives, of course, unity). Thus

$$\frac{\partial}{\partial t}\int f(\mathbf{x}) q_{\{\mathbf{Z}\}}(\mathbf{x}, t|\mathbf{y}, t_0)d\mathbf{x} = \lim_{\Delta t \to 0}\frac{1}{\Delta t}\left\{\int\int_{|\mathbf{x}-\mathbf{z}|<\varepsilon}\left[\sum_i (x_i - z_i)\frac{\partial f(\mathbf{z})}{\partial z_i} + \right.\right.$$

$$\left.\frac{1}{2}\sum_{i,j}(x_i - z_i)(x_j - z_j)\frac{\partial^2 f(\mathbf{z})}{\partial z_i \partial z_j}\right] q_{\{\mathbf{Z}\}}(\mathbf{x}, t+\Delta t|\mathbf{z}, t) q_{\{\mathbf{Z}\}}(\mathbf{z}, t|\mathbf{y}, t_0)d\mathbf{x}d\mathbf{z} +$$

$$\int\int_{|\mathbf{x}-\mathbf{z}|<\varepsilon} \mathrm{O}\Big(|\mathbf{x}-\mathbf{z}|^3\Big) q_{\{\mathbf{Z}\}}(\mathbf{x}, t+\Delta t|\mathbf{z}, t) q_{\{\mathbf{Z}\}}(\mathbf{z}, t|\mathbf{y}, t_0)d\mathbf{x}d\mathbf{z} +$$

$$\int\int_{|\mathbf{x}-\mathbf{z}|<\varepsilon} f(\mathbf{z}) q_{\{\mathbf{Z}\}}(\mathbf{x}, t+\Delta t|\mathbf{z}, t) q_{\{\mathbf{Z}\}}(\mathbf{z}, t|\mathbf{y}, t_0)d\mathbf{x}d\mathbf{z} +$$

$$\int\int_{|\mathbf{x}-\mathbf{z}|\geq\varepsilon} f(\mathbf{x}) q_{\{\mathbf{Z}\}}(\mathbf{x}, t+\Delta t|\mathbf{z}, t) q_{\{\mathbf{Z}\}}(\mathbf{z}, t|\mathbf{y}, t_0)d\mathbf{x}d\mathbf{z} -$$

$$\left.\int\int f(\mathbf{z}) q_{\{\mathbf{Z}\}}(\mathbf{x}, t+\Delta t|\mathbf{z}, t) q_{\{\mathbf{Z}\}}(\mathbf{z}, t|\mathbf{y}, t_0)d\mathbf{x}d\mathbf{z}\right\} \qquad (2.19)$$

Since the uniform convergence in $\mathbf{z}$ has been assumed for the limits (2.11) and (2.12), the order of limiting operation and integration over $\mathbf{z}$ in first two integrals of (2.19) can be interchanged. For the limits in these integrals we use (2.11) and (2.12). The second integral involves all derivative moments above the second order. These all vanish for a continuous Markov process as $\Delta t \to 0$. Finally, we interchange the variables $\mathbf{x}$ and $\mathbf{z}$ in the fourth integral, and we divide the domain of integration in the last integral into two subdomains: $|\mathbf{x}-\mathbf{z}| < \varepsilon$ and $|\mathbf{x}-\mathbf{z}| \geq \varepsilon$. Thus

$$\frac{\partial}{\partial t}\int f(\mathbf{x}) q_{\{\mathbf{Z}\}}(\mathbf{x}, t|\mathbf{y}, t_0)d\mathbf{x} =$$

$$\int\left[\sum_i C_i(\mathbf{z}, t)\frac{\partial f(\mathbf{z})}{\partial z_i} + \frac{1}{2}\sum_{i,j} D_{ij}(\mathbf{z}, t)\frac{\partial^2 f(\mathbf{z})}{\partial z_i \partial z_j}\right] q_{\{\mathbf{Z}\}}(\mathbf{z}, t|\mathbf{y}, t_0)d\mathbf{z} +$$

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\{\int\int_{|\mathbf{x}-\mathbf{z}|<\varepsilon} f(\mathbf{z})q_{\{\mathbf{Z}\}}(\mathbf{x},t+\Delta t|\mathbf{z},t)q_{\{\mathbf{Z}\}}(\mathbf{z},t|\mathbf{y},t_0)d\mathbf{x}d\mathbf{z}+\right.$$
$$\int\int_{|\mathbf{x}-\mathbf{z}|\geq\varepsilon} f(\mathbf{z})q_{\{\mathbf{Z}\}}(\mathbf{z},t+\Delta t|\mathbf{x},t)q_{\{\mathbf{Z}\}}(\mathbf{x},t|\mathbf{y},t_0)d\mathbf{x}d\mathbf{z}-$$
$$\int\int_{|\mathbf{x}-\mathbf{z}|\geq\varepsilon} f(\mathbf{z})q_{\{\mathbf{Z}\}}(\mathbf{x},t+\Delta t|\mathbf{z},t)q_{\{\mathbf{Z}\}}(\mathbf{z},t|\mathbf{y},t_0)d\mathbf{x}d\mathbf{z}-$$
$$\left.\int\int_{|\mathbf{x}-\mathbf{z}|<\varepsilon} f(\mathbf{z})q_{\{\mathbf{Z}\}}(\mathbf{x},t+\Delta t|\mathbf{z},t)q_{\{\mathbf{Z}\}}(\mathbf{z},t|\mathbf{y},t_0)d\mathbf{x}d\mathbf{z}\right\} \tag{2.20}$$

We note that the integrals over $|\mathbf{x}-\mathbf{z}|<\varepsilon$ yield zero $\varepsilon\to 0$. By virtue of the uniform convergence in $\mathbf{x}$ and $\mathbf{z}$ we interchange the order of the limiting operation and integration in the remaining integrals over $|\mathbf{x}-\mathbf{z}|\geq\varepsilon$. Introducing the jump probability intensity function (2.10) we have

$$\frac{\partial}{\partial t}\int f(\mathbf{z})q_{\{\mathbf{Z}\}}(\mathbf{z},t|\mathbf{y},t_0)d\mathbf{z}=$$
$$\int\left[\sum_i C_i(\mathbf{z},t)\frac{\partial f(\mathbf{z})}{\partial z_i}+\frac{1}{2}\sum_{i,j}D_{ij}(\mathbf{z},t)\frac{\partial^2 f(\mathbf{z})}{\partial z_i\partial z_j}\right]q_{\{\mathbf{Z}\}}(\mathbf{z},t|\mathbf{y},t_0)d\mathbf{z}+$$
$$\int\int f(\mathbf{z})\Big(J_{\{\mathbf{Z}\}}(\mathbf{z}|\mathbf{x},t)q_{\{\mathbf{Z}\}}(\mathbf{x},t|\mathbf{y},t_0)-J_{\{\mathbf{Z}\}}(\mathbf{x}|\mathbf{z},t)q_{\{\mathbf{Z}\}}(\mathbf{z},t|\mathbf{y},t_0)\Big)d\mathbf{x}d\mathbf{z} \tag{2.21}$$

where the last innermost integral need not exist in ordinary Riemann sense but it should merely be regarded as a Cauchy principal value integral defined as

$$\int J_{\{\mathbf{Z}\}}(\mathbf{z}|\mathbf{x},t)q_{\{\mathbf{Z}\}}(\mathbf{x},t|\mathbf{y},t_0)d\mathbf{x}=$$
$$\lim_{\varepsilon\to 0}\int_{|\mathbf{x}-\mathbf{z}|\geq\varepsilon} J_{\{\mathbf{Z}\}}(\mathbf{z}|\mathbf{x},t)q_{\{\mathbf{Z}\}}(\mathbf{x},t|\mathbf{y},t_0)d\mathbf{x} \tag{2.22}$$

Such a definition is required to assure the existence of the integrals in the equation (2.21), in order for this equation to be meaningful. Notice that the jump probability intensity function $J_{\{\mathbf{Z}\}}(\mathbf{x}|\mathbf{z},t)$ is only defined for $|\mathbf{x}-\mathbf{z}|\geq\varepsilon$, hence for $\mathbf{x}\neq\mathbf{z}$. It may tend to infinity as $\mathbf{x}\to\mathbf{z}$, consequently as it appears in the integrand the integral becomes infinite. If $q_{\{\mathbf{Z}\}}(\mathbf{x},t|\mathbf{y},t_0)$ is continuous and once differentiable, the integrals in (2.21) exist in the sense of principal value.

Integrating by parts the integrands of the first integral on the right-hand side of (2.21) yields

$$\frac{\partial}{\partial t}\int f(\mathbf{z})q_{\{\mathbf{Z}\}}(\mathbf{z},t|\mathbf{y},t_0)d\mathbf{z}=$$

$$\int f(\mathbf{z}) \left\{ -\sum_i \frac{\partial}{\partial z_i} \left(C_i(\mathbf{z},t) q_{\{\mathbf{Z}\}}(\mathbf{z},t|\mathbf{y},t_0) \right) + \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial z_i \partial z_j} \left(D_{ij}(\mathbf{z},t) q_{\{\mathbf{Z}\}}(\mathbf{z},t|\mathbf{y},t_0) \right) \right.$$
$$\left. + \int \left[J_{\{\mathbf{Z}\}}(\mathbf{z}|\mathbf{x},t) q_{\{\mathbf{Z}\}}(\mathbf{x},t|\mathbf{y},t_0) - J_{\{\mathbf{Z}\}}(\mathbf{x}|\mathbf{z},t) q_{\{\mathbf{Z}\}}(\mathbf{z},t|\mathbf{y},t_0) \right] d\mathbf{x} \right\} d\mathbf{z} \qquad (2.23)$$

where it has been assumed that the surface terms vanish.

The equation (2.23) must hold for any function $f(\mathbf{z})$. Hence the following forward integro-differential Chapman-Kolmogorov equation is arrived at

$$\frac{\partial}{\partial t} q_{\{\mathbf{Z}\}}(\mathbf{z},t|\mathbf{y},t_0) = \mathcal{K}_{\mathbf{z},t}\left[q_{\{\mathbf{Z}\}}(\mathbf{z},t \mid \mathbf{y},t_0) \right] \qquad (2.24)$$

$$\mathcal{K}_{\mathbf{z},t}\left[q_{\{\mathbf{Z}\}}(\mathbf{z},t \mid \mathbf{y},t_0) \right] =$$
$$-\sum_i \frac{\partial}{\partial z_i} [C_i(\mathbf{z},t) q_{\{\mathbf{Z}\}}(\mathbf{z},t|\mathbf{y},t_0)] + \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial z_i \partial z_j} [D_{ij}(\mathbf{z},t) q_{\{\mathbf{Z}\}}(\mathbf{z},t|\mathbf{y},t_0)] +$$
$$\int [J_{\{\mathbf{Z}\}}(\mathbf{z}|\mathbf{x},t) q_{\{\mathbf{Z}\}}(\mathbf{x},t|\mathbf{y},t_0) - J_{\{\mathbf{Z}\}}(\mathbf{x}|\mathbf{z},t) q_{\{\mathbf{Z}\}}(\mathbf{z},t|\mathbf{y},t_0)] d\mathbf{x} \qquad (2.25)$$

Equation (2.24) is called a master equation in physics. $\mathcal{K}_{\mathbf{z},t}[...]$ is the forward integro-differential Chapman-Kolmogorov operator. This equation must be solved with relevant initial and boundary conditions which will be specified in section 2.1.4.

The boundary conditions on the surface ∂S_t at the time t are specified by the vanishing of the surface terms at the derivation of (2.23). These become

$$\sum_{i=1}^{n} \int_{\partial S_t} f(\mathbf{z}) n_i(\mathbf{z},t) \left[C_i(\mathbf{z},t) q_{\{\mathbf{Z}\}}(\mathbf{z},t|\mathbf{y},t_0) - \frac{1}{2} \sum_{j=1}^{n} \frac{\partial}{\partial z_j} \left(D_{ji}(\mathbf{z},t) q_{\{\mathbf{Z}\}}(\mathbf{z},t|\mathbf{y},t_0) \right) \right] d\mathbf{z}$$
$$+ \frac{1}{2} \sum_{j=1}^{n} \int_{\partial S_t} \frac{\partial f(\mathbf{z})}{\partial z_j} n_i(\mathbf{z},t) D_{ij}(\mathbf{z},t) q_{\{\mathbf{Z}\}}(\mathbf{z},t|\mathbf{y},t_0) d\mathbf{z} = 0 \qquad (2.26)$$

where $n_i(\mathbf{z},t)$ signifies the outward directed unit vector at the position $\mathbf{z}$ at the time t on the surface ∂S_t. We may restrict the variations so that $\frac{\partial}{\partial z_j} f(\mathbf{z}) = f(\mathbf{z}) \equiv 0$ on some part $\partial S_t^{(1)}$ of the surface, whereas arbitrary variations are admitted on the remaining part $\partial S_t^{(2)}$. On $\partial S_t^{(2)}$ (2.26) can then only be fulfilled, if for all $\mathbf{z} \in \partial S_t^{(2)}$ it holds

$$\sum_{i=1}^{n} n_i(\mathbf{z},t) \left[C_i(\mathbf{z},t) q_{\{\mathbf{Z}\}}(\mathbf{z},t|\mathbf{y},t_0) - \frac{1}{2} \sum_{j=1}^{n} \frac{\partial}{\partial z_j} \left(D_{ji}(\mathbf{z},t) q_{\{\mathbf{Z}\}}(\mathbf{z},t|\mathbf{y},t_0) \right) \right] = 0 \qquad (2.27)$$

$$\sum_{i=1}^{n} n_i(\mathbf{z},t)D_{ij}(\mathbf{z},t)q_{\{\mathbf{Z}\}}(\mathbf{z},t|\mathbf{y},t_0) = 0 \tag{2.28}$$

Implicitly it has been assumed, at the derivation of (2.25), that for any $\mathbf{z} \in \partial S_t^c$, $\mathbf{x} \in S_t$

$$J_{\{\mathbf{Z}\}}(\mathbf{z}|\mathbf{x},t) = 0 \tag{2.29}$$

The condition (2.29) states that no finite jumps out of the domain S_t take place. Hence the domain S_t can only be left by convection or diffusion.

The boundary ∂S_t is called inaccessible if it cannot be reached in a finite time. This means that for any $\mathbf{z} \in \partial S_t$, $\mathbf{y} \in S_{t_0}$ one has

$$q_{\{\mathbf{Z}\}}(\mathbf{z},t|\mathbf{y},t_0) = 0 \tag{2.30}$$

Obviously (2.26) is fulfilled in this case.

2.1.3 Backward integro-differential Chapman-Kolmogorov equation

Let us now derive the backward form of the integro-differential Chapman-Kolmogorov equation. We will examine the evolution in time of the transition probability density function $q_{\{\mathbf{Z}\}}(\mathbf{x},t_1|\mathbf{z},t)$ with respect to the initial (backward) variables $\mathbf{z},t$, where $\mathbf{x}$ is the terminal state and t_1 is the terminal time. Thus

$$\frac{\partial}{\partial t}q_{\{\mathbf{Z}\}}(\mathbf{x},t_1|\mathbf{z},t) = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\Big(q_{\{\mathbf{Z}\}}(\mathbf{x},t_1|\mathbf{z},t+\Delta t) - q_{\{\mathbf{Z}\}}(\mathbf{x},t_1|\mathbf{z},t)\Big) \tag{2.31}$$

The first step is to represent the second term in form of the Chapman-Kolmogorov equation

$$q_{\{\mathbf{Z}\}}(\mathbf{x},t_1|\mathbf{z},t) = \int q_{\{\mathbf{Z}\}}(\mathbf{x},t_1|\mathbf{y},t+\Delta t)q_{\{\mathbf{Z}\}}(\mathbf{y},t+\Delta t|\mathbf{z},t)d\mathbf{y} \tag{2.32}$$

Further, the first term of (2.31) is multiplied by $q_{\{\mathbf{Z}\}}(\mathbf{y},t+\Delta t|\mathbf{z},t)$ and integrated over $\mathbf{y}$, the latter being an admissible operation as it gives 1. Thus

$$\frac{\partial}{\partial t}q_{\{\mathbf{Z}\}}(\mathbf{x},t_1|\mathbf{z},t) =$$

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int q_{\{\mathbf{Z}\}}(\mathbf{y},t+\Delta t|\mathbf{z},t)\Big(q_{\{\mathbf{Z}\}}(\mathbf{x},t_1|\mathbf{z},t+\Delta t) - q_{\{\mathbf{Z}\}}(\mathbf{x},t_1|\mathbf{y},t+\Delta t)\Big)d\mathbf{y} \tag{2.33}$$

Under the assumption of existence of all relevant derivatives, and of the continuity of $q_{\{\mathbf{Z}\}}(\mathbf{x}, t_1|\mathbf{z}, t)$ and its boundedness in $\mathbf{x}, t, t_1$ for some range $t_1 - t > \delta > 0$, the expression (2.33) may be recast into the form of

$$\frac{\partial}{\partial t} q_{\{\mathbf{Z}\}}(\mathbf{x}, t_1|\mathbf{z}, t) =$$

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int q_{\{\mathbf{Z}\}}(\mathbf{y}, t + \Delta t|\mathbf{z}, t)\Big(q_{\{\mathbf{Z}\}}(\mathbf{x}, t_1|\mathbf{z}, t) - q_{\{\mathbf{Z}\}}(\mathbf{x}, t_1|\mathbf{y}, t)\Big) d\mathbf{y} \tag{2.34}$$

Proceeding as before let us divide the domain of integration into two subdomains $|\mathbf{y} - \mathbf{z}| < \varepsilon$ and $|\mathbf{y} - \mathbf{z}| \geq \varepsilon$, thus

$$\frac{\partial}{\partial t} q_{\{\mathbf{Z}\}}(\mathbf{x}, t_1|\mathbf{z}, t) =$$

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int\limits_{|\mathbf{y}-\mathbf{z}|<\varepsilon} q_{\{\mathbf{Z}\}}(\mathbf{y}, t + \Delta t|\mathbf{z}, t)\Big(q_{\{\mathbf{Z}\}}(\mathbf{x}, t_1|\mathbf{z}, t) - q_{\{\mathbf{Z}\}}(\mathbf{x}, t_1|\mathbf{y}, t)\Big) d\mathbf{y} +$$

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int\limits_{|\mathbf{y}-\mathbf{z}|\geq\varepsilon} q_{\{\mathbf{Z}\}}(\mathbf{y}, t + \Delta t|\mathbf{z}, t)\Big(q_{\{\mathbf{Z}\}}(\mathbf{x}, t_1|\mathbf{z}, t) - q_{\{\mathbf{Z}\}}(\mathbf{x}, t_1|\mathbf{y}, t)\Big) d\mathbf{y} \tag{2.35}$$

Recalling again the assumption about the existence of all relevant derivatives of $q_{\{\mathbf{Z}\}}(\mathbf{x}, t_1|\mathbf{z}, t)$ we can use the following Taylor expansion for the negative term in the first integral

$$q_{\{\mathbf{Z}\}}(\mathbf{x}, t_1|\mathbf{y}, t) = q_{\{\mathbf{Z}\}}(\mathbf{x}, t_1|\mathbf{z}, t) + \sum_i (y_i - z_i) \frac{\partial q_{\{\mathbf{Z}\}}(\mathbf{x}, t_1|\mathbf{z}, t)}{\partial z_i} +$$

$$\frac{1}{2} \sum_{i,j} (y_i - z_i)(y_j - z_j) \frac{\partial^2 q_{\{\mathbf{Z}\}}(\mathbf{x}, t_1|\mathbf{z}, t)}{\partial z_i \partial z_j} + \mathrm{O}\left(|\mathbf{y} - \mathbf{z}|^3\right) \tag{2.36}$$

Next we insert this expansion into (2.35), which yields

$$\frac{\partial}{\partial t} q_{\{\mathbf{Z}\}}(\mathbf{x}, t_1|\mathbf{z}, t) =$$

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int\limits_{|\mathbf{y}-\mathbf{z}|<\varepsilon} q_{\{\mathbf{Z}\}}(\mathbf{y}, t + \Delta t|\mathbf{z}, t)\Big[- \sum_i (y_i - z_i) \frac{\partial q_{\{\mathbf{Z}\}}(\mathbf{x}, t_1|\mathbf{z}, t)}{\partial z_i} -$$

$$\frac{1}{2} \sum_{i,j} (y_i - z_i)(y_j - z_j) \frac{\partial^2 q_{\{\mathbf{Z}\}}(\mathbf{x}, t_1|\mathbf{z}, t)}{\partial z_i \partial z_j}\Big] d\mathbf{y} +$$

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int\limits_{|\mathbf{y}-\mathbf{z}|\geq\varepsilon} q_{\{\mathbf{Z}\}}(\mathbf{y}, t + \Delta t|\mathbf{z}, t)\Big(q_{\{\mathbf{Z}\}}(\mathbf{x}, t_1|\mathbf{z}, t) - q_{\{\mathbf{Z}\}}(\mathbf{x}, t_1|\mathbf{y}, t)\Big) d\mathbf{y} \tag{2.37}$$

The first integral yields the derivate moments (cf. the expressions (2.11) and (2.12)). In (2.37) the remainder has been omitted, as it would only give the higher-order derivate moments, which in the case of a continuous motion, vanish. After interchanging the order of limiting operation and integration in the second integral the jump probability intensity function $J_{\{\mathbf{Z}\}}(\mathbf{y}|\mathbf{z},t)$ is obtained, cf. (2.10).

Finally, the following equation is arrived at

$$\frac{\partial}{\partial t}q_{\{\mathbf{Z}\}}(\mathbf{x},t_1|\mathbf{z},t) + \mathcal{K}^T_{\mathbf{z},t}\left[q_{\{\mathbf{Z}\}}(\mathbf{x},t_1 \mid \mathbf{z},t)\right] = 0 \tag{2.38}$$

$$\mathcal{K}^T_{\mathbf{z},t}\left[q_{\{\mathbf{Z}\}}(\mathbf{x},t_1 \mid \mathbf{z},t)\right] =$$

$$\sum_i C_i(\mathbf{z},t)\frac{\partial q_{\{\mathbf{Z}\}}(\mathbf{x},t_1|\mathbf{z},t)}{\partial z_i} + \frac{1}{2}\sum_{i,j} D_{ij}(\mathbf{z},t)\frac{\partial^2 q_{\{\mathbf{Z}\}}(\mathbf{x},t_1|\mathbf{z},t)}{\partial z_i \partial z_j} -$$

$$\int J_{\{\mathbf{Z}\}}(\mathbf{y}|\mathbf{z},t)\big(q_{\{\mathbf{Z}\}}(\mathbf{x},t_1|\mathbf{z},t) - q_{\{\mathbf{Z}\}}(\mathbf{x},t_1|\mathbf{y},t)\big)d\mathbf{y} \tag{2.39}$$

(2.38) is the backward integro-differential Chapman-Kolmogorov equation, or just a backward counterpart of the master equation (2.24). $\mathcal{K}^T_{\mathbf{z},t}[\cdot]$ signifies the backward integro-differential Chapman-Kolmogorov operator.

2.1.4 Initial and boundary conditions for the forward and backward integro-differential Chapman-Kolmogorov equations

The forward and backward Chapman-Kolmogorov integro-differential equations must be solved with proper boundary conditions, which will be derived in this section based on a requirement that $q_{\{\mathbf{Z}\}}(\mathbf{z},t|\mathbf{x},t_0)$ should fulfill both equations.

In figure 2.1 the sample path is shown for the single-degree-of-freedom non-hysteretic oscillator subjected to a combined Wiener process and jump process excitation. The domain S_t of the state vector $\mathbf{Z}^T(t) = [X(t), \dot{X}(t)]$ is the so-called double barrier domain, where the displacement component $X(t)$ is confined to the interval $]a,b[$, whereas the velocity $\dot{X}$ can take values from $]-\infty,\infty[$.

In general it is assumed that the n-dimensional state vector $\mathbf{Z}(t)$ is restricted to some domain $S_t \subseteq R^n$ which may depend on time. In figure 2.1 the domain is given by $S_t = \{(x,\dot{x})|a < x < b \quad \wedge \quad -\infty < \dot{x} < \infty\}$. The domain is time varying if the boundaries a and b depend on time e.g. due to the strain or strength hardening effects.

The surface (boundary) of the domain is denoted ∂S_t. It may be divided into the accessible part $\partial S_t^{(a)}$, which can be reached in a finite transition time and the non-accessible part $\partial S_t^{(2)}$, which can only be reached after infinitely long time intervals. These parts are defined as follows

$$\partial S_t^{(a)} = \left\{\mathbf{z} \in \partial S_t \mid \forall\, \mathbf{x} \in S_{t_0} \;:\; q_{\{\mathbf{Z}\}}(\mathbf{z},t \mid \mathbf{x},t_0) > 0\right\} \tag{2.40}$$

$$\partial S_t^{(2)} = \left\{\mathbf{z} \in \partial S_t \mid \forall\, \mathbf{x} \in S_{t_0} \;:\; q_{\{\mathbf{Z}\}}(\mathbf{z},t \mid \mathbf{x},t_0) = 0\right\} \tag{2.41}$$

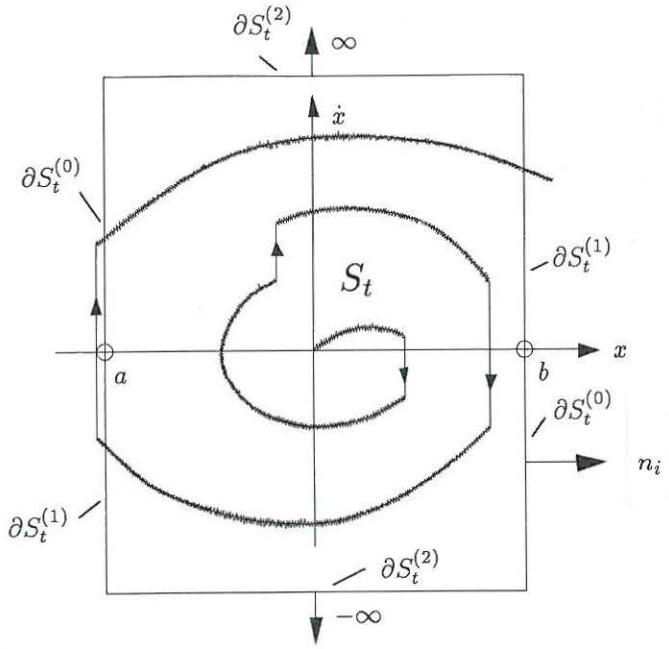

Figure 2.1. Sample path of the state vector of SDOF non-hysteretic oscillator subjected to Wiener process and jump process excitation.

In the example shown in figure 2.1 the accessible and non-accessible parts of the boundary are given as $\partial S_t^{(a)} = \{(x,\dot{x})|(x = a \quad \vee \quad x = b) \quad \wedge \quad -\infty < \dot{x} < \infty\}$ and $\partial S_t^{(2)} = \{(x,\dot{x})|a < x < b \quad \wedge \quad (\dot{x} = -\infty \quad \vee \quad \dot{x} = \infty)\}$.

For the system shown in figure 2.1 a jump of the generating source process $\{V(t),\ t \in [0,\infty[\}$ causes a discontinuity (jump) in the velocity component of the state vector, cf. (1.90), (1.91). Especially, the jumps close to the accessible surface $\partial S_t^{(a)}$ are tangential. Hence no jumps are possible out of the domain. Generally this is assumed to be true, i.e. the jump probability intensity function fulfils $J_{\{\mathbf{Z}\}}(\mathbf{z} \mid \mathbf{y}, t) = 0$ for all $\mathbf{y} \in S_t$ and $\mathbf{z} \in S_t^c$. Any finite jump at $\mathbf{z}$ close to the accessible part of the surface $\partial S_t^{(a)}$ then takes place in the tangential direction.

Hence, the flux of probability mass through the accessible surface $\partial S_t^{(a)}$ is caused totally by the convection and diffusion components. The abbreviates $q^{(0)}(\mathbf{z},t) \equiv q_{\{\mathbf{Z}\}}(\mathbf{z},t \mid \mathbf{x},t_0)$ and $q^{(1)}(\mathbf{z},t) \equiv q_{\{\mathbf{Z}\}}(\mathbf{y},t_1 \mid \mathbf{z},t)$ are introduced. Application of the divergence theorem then provides

$$\int_{t_0}^{t_1}\int_{S_t} q^{(1)}(\mathbf{z},t)\Big(\frac{\partial q^{(0)}(\mathbf{z},t)}{\partial t} - \mathcal{K}_{\mathbf{z},t}\big[q^{(0)}(\mathbf{z},t)\big]\Big)d\mathbf{z}dt =$$

$$-\int_{t_0}^{t_1}\int_{S_t} q^{(0)}(\mathbf{z},t)\Big(\frac{\partial q^{(1)}(\mathbf{z},t)}{\partial t} + \mathcal{K}^T_{\mathbf{z},t}\big[q^{(1)}(\mathbf{z},t)\big]\Big)d\mathbf{z}dt + R\big[q^{(0)}(\mathbf{z},t), q^{(1)}(\mathbf{z},t)\big] \quad (2.42)$$

where $\mathcal{K}_{\mathbf{z},t}[\cdot]$ and $\mathcal{K}^T_{\mathbf{z},t}[\cdot]$ are the forward and backward Kolmogorov operators as given by (2.25) and (2.39), and

$$R\big[q^{(0)}(\mathbf{z},t), q^{(1)}(\mathbf{z},t)\big] =$$

$$\int_{S_{t_1}} q_{\{\mathbf{Z}\}}(\mathbf{y},t_1 \mid \mathbf{z},t_1) q_{\{\mathbf{Z}\}}(\mathbf{z},t_1 \mid \mathbf{x},t_0) d\mathbf{z} - \int_{S_{t_0}} q_{\{\mathbf{Z}\}}(\mathbf{y},t_1 \mid \mathbf{z},t_0) q_{\{\mathbf{Z}\}}(\mathbf{z},t_0 \mid \mathbf{x},t_0) d\mathbf{z} +$$

$$\sum_{i=1}^{n} \int_{t_0}^{t_1} \int_{\partial S_t} n_i(\mathbf{z},t) \Big[\Big(C_i(\mathbf{z},t) q^{(0)}(\mathbf{z},t) - \sum_{j=1}^{n} \frac{1}{2} \frac{\partial}{\partial z_j} \big(D_{ij}(\mathbf{z},t) q^{(0)}(\mathbf{z},t) \big) \Big) q^{(1)}(\mathbf{z},t) +$$

$$\sum_{j=1}^{n} q^{(0)}(\mathbf{z},t) \frac{1}{2} D_{ij}(\mathbf{z},t) \frac{\partial}{\partial z_j} q^{(1)}(\mathbf{z},t) \Big] da_t dt \tag{2.43}$$

where $n_i(\mathbf{z},t)$ signifies the unit normal vector in the outward direction of the surface ∂S_t.

If $q^{(0)}(\mathbf{z},t) \equiv q_{\{\mathbf{Z}\}}(\mathbf{z},t \mid \mathbf{x},t_0)$ and $q^{(1)}(\mathbf{z},t) \equiv q_{\{\mathbf{Z}\}}(\mathbf{y},t_1 \mid \mathbf{z},t)$ are assumed to fulfil the forward and backward Kolmogorov equations (2.24) and (2.38) throughout S_t, the left-hand side and the first term on the right-hand side of (2.42) cancel. Hence, $R\big[q^{(0)}(\mathbf{z},t), q^{(1)}(\mathbf{z},t)\big] \equiv 0$. The initial time t_0 and the terminal time t_1 may be varied. From (2.43) the necessity of the following initial and boundary conditions for $q_{\{\mathbf{Z}\}}(\mathbf{z},t \mid \mathbf{x},t_0)$ and $q_{\{\mathbf{Z}\}}(\mathbf{y},t_1 \mid \mathbf{z},t)$ is then deduced

$$\forall\, \mathbf{x},\mathbf{z} \in S_{t_0} : \quad q_{\{\mathbf{Z}\}}(\mathbf{z},t_0 \mid \mathbf{x},t_0) = \delta(\mathbf{z}-\mathbf{x}) \tag{2.44}$$

$$\forall\, \mathbf{y},\mathbf{z} \in S_{t_1} : \quad q_{\{\mathbf{Z}\}}(\mathbf{y},t_1 \mid \mathbf{z},t_1) = \delta(\mathbf{z}-\mathbf{y}) \tag{2.45}$$

$$\forall\, t \in]t_0,t_1[\ ,\ \mathbf{x} \in S_{t_0}\ ,\ \mathbf{z} \in \partial S_t^{(2)} : \quad q^{(0)}(\mathbf{z},t) = q_{\{\mathbf{Z}\}}(\mathbf{z},t \mid \mathbf{x},t_0) = 0 \tag{2.46}$$

$$\forall\, t \in]t_0,t_1[\ ,\ \mathbf{y} \in S_{t_1}\ ,\ \mathbf{z} \in \partial S_t^{(2)} : \quad q^{(1)}(\mathbf{z},t) = q_{\{\mathbf{Z}\}}(\mathbf{y},t_1 \mid \mathbf{z},t) = 0 \tag{2.47}$$

$$\forall\, t \in]t_0,t_1[\ ,\ \mathbf{x} \in S_{t_0}\ ,\ \mathbf{y} \in S_{t_1}\ ,\ \mathbf{z} \in \partial S_t :$$

$$\sum_{i=1}^{n} \int_{\partial S_t^{(a)}} n_i(\mathbf{z},t) \Big[\Big(C_i(\mathbf{z},t) q^{(0)}(\mathbf{z},t) - \frac{1}{2} \sum_{j=1}^{n} \frac{\partial}{\partial z_j} \big(D_{ij}(\mathbf{z},t) q^{(0)}(\mathbf{z},t) \big) \Big) q^{(1)}(\mathbf{z},t) +$$

$$\sum_{j=1}^{n} q^{(0)}(\mathbf{z},t) \frac{1}{2} D_{ij}(\mathbf{z},t) \frac{\partial}{\partial z_j} q^{(1)}(\mathbf{z},t) \Big] da_t = 0 \tag{2.48}$$

The conditions (2.44) and (2.45) specify the initial and terminal conditions to be used for the forward and backward Chapman-Kolmogorov integro-differential equations, respectively.

The condition (2.46) signifies that the probability density for arriving at a boundary point $\mathbf{z}$ on the non-accessible boundary $\partial S_t^{(2)}$ from an internal point $\mathbf{x}$ in the domain S_{t_0} is zero for the finite transition times $t - t_0$. Similarly the condition (2.47) states that the transitions from a boundary point $\mathbf{z}$ at ∂S_t to an internal point $\mathbf{y}$ in S_{t_1} are zero for the finite transition times $t_1 - t$.

Due to these conditions the surface integral in (2.48) can be confined to the accessible part $\partial S_t^{(a)}$ of the boundary. (2.48) is the necessary condition which must be fulfilled by the forward and backward differential Chapman-Kolmogorov equations in combination. For example, (2.48) may be fulfilled by the forward equation on some part of $\partial S_t^{(a)}$ and by the backward equation on the remaining part.

Consider two functions $u(\mathbf{z})$ and $v(\mathbf{z})$ defined on S_t. If these functions in combination fulfil the boundary conditions (2.46), (2.47) and (2.48), it follows from the indicated derivation that

$$\int_{S_t} u(\mathbf{z}) \mathcal{K}_{\mathbf{z},t}[v(\mathbf{z})] d\mathbf{z} = \int_{S_t} v(\mathbf{z}) \mathcal{K}^T_{\mathbf{z},t}[u(\mathbf{z})] d\mathbf{z} \tag{2.49}$$

This means that (2.46), (2.48) and (2.49) are the necessary conditions in order for $\mathcal{K}_{\mathbf{z},t}[\cdot]$ and $\mathcal{K}^T_{\mathbf{z},t}[\cdot]$ to be mutually adjoint operators.

$\partial S_t^{(a)}$ may be further divided into the entrance part, $\partial S_t^{(0)}$, and the exit part, $\partial S_t^{(1)}$, which in case of the indicated jump condition are defined in the same way as for diffusion processes, Fichera (1960)

$$\partial S_t^{(0)} = \Big\{ \mathbf{z} \in \partial S_t^{(a)} \Big| \sum_{i=1}^{n} n_i(\mathbf{z},t) \Big(C_i(\mathbf{z},t) - \frac{1}{2} \frac{\partial}{\partial z_j} D_{ij}(\mathbf{z},t) \Big) < 0 \Big\} \tag{2.50}$$

$$\partial S_t^{(1)} = \Big\{ \mathbf{z} \in \partial S_t^{(a)} \Big| \sum_{i=1}^{n} n_i(\mathbf{z},t) \Big(C_i(\mathbf{z},t) - \frac{1}{2} \frac{\partial}{\partial z_j} D_{ij}(\mathbf{z},t) \Big) > 0 \Big\} \tag{2.51}$$

(2.50) and (2.51) cover all $\partial S_t^{(a)}$ except at certain isolated points, where the probability current is tangential to the surface. Further, the boundary may be divided into the degenerated part, where $\sum_{i,j} D_{ij}(\mathbf{z},t) n_i(\mathbf{z},t) n_j(\mathbf{z},t) = 0$, and the non-degenerated part, where $\sum_{i,j} D_{ij}(\mathbf{z},t) n_i(\mathbf{z},t) n_j(\mathbf{z},t) \neq 0$, Fichera (1960). For the present dynamic system and the indicated jump condition, all accessible parts of the boundary are degenerated, and only the non-accessible parts can be non-degenerated.

For the system shown in figure 2.1 the entrance part is $\partial S_t^{(0)} = \{(x,\dot{x}) | (x = a \wedge \dot{x} > 0) \vee (x = b \wedge \dot{x} < 0)\}$ and the exit part is $\partial S_t^{(1)} = \{(x,\dot{x}) | (x = a \wedge \dot{x} < 0) \vee (x = b \wedge \dot{x} > 0)\}$. The surface parts have been indicated in the figure along with the non-accessible part $\partial S_t^{(2)}$.

2.2 Relation of forward and backward integro-differential Chapman-Kolmogorov equation to equations for dynamical systems

Hitherto nothing has been said about the relation of equations (2.24) and (2.38) to the dynamical Markov system governed by equation (1.105). In this section such a relation is discussed, i.e. the functions $\mathbf{C}(\mathbf{z}(t),t)$ and $\mathbf{D}(\mathbf{z}(t),t)$ and the jump probability intensity function $J_{\{\mathbf{Z}\}}(\mathbf{x}|\mathbf{z},t)$ are related to the quantities entering the equation (1.105).

Upon insertion of (1.105) into (2.13) and (2.14) and observing that only the continuous part, i.e. $d\mathbf{Z}(t) = \mathbf{c}(\mathbf{Z}(t),t)dt + \mathbf{d}(\mathbf{Z}(t),t)d\mathbf{W}(t)$ is to be inserted into these expressions, one has

$$C_i(\mathbf{z},t) = \lim_{\Delta t\to 0}\frac{1}{\Delta t}E\Big[\Delta Z_i(t)|\mathbf{Z}(t)=\mathbf{z}\Big] =$$

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}E\left[c_i(\mathbf{z},t)\Delta t + \sum_{\alpha=1}^{m} d_{i\alpha}(\mathbf{z},t)\Delta W_\alpha(t) + \mathrm{O}_i(\Delta t^2)\Big|\mathbf{Z}(t)=\mathbf{z}\right] = c_i(\mathbf{z},t) \quad (2.52)$$

$$D_{ij}(\mathbf{z},t) = \lim_{\Delta t\to 0}\frac{1}{\Delta t}E\Big[\Delta Z_i(t)\Delta Z_j(t)|\mathbf{Z}(t)=\mathbf{z}\Big] =$$

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}E\left[\left(c_i(\mathbf{z},t)\Delta t + \sum_{\alpha=1}^{m} d_{i\alpha}(\mathbf{z},t)\Delta W_\alpha(t) + \mathrm{O}_i(\Delta t^2)\right)\times\right.$$

$$\left.\left(c_j(\mathbf{z},t)\Delta t + \sum_{\beta=1}^{m} d_{j\beta}(\mathbf{z},t)\Delta W_\beta(t) + \mathrm{O}_j(\Delta t^2)\right)\Big|\mathbf{Z}(t)=\mathbf{z}\right] =$$

$$\sum_{\alpha=1}^{m} d_{i\alpha}(\mathbf{z}(t),t)\,d_{j\alpha}(\mathbf{z}(t),t) \quad (2.53)$$

where $\mathrm{O}_i(\Delta t^2)$ is a component order symbol. Further the incremental properties (1.12) for the Wiener process and the Itô interpretation of the stochastic differential equation have been used.

In order to derive the jump probability intensity function $J_{\{\mathbf{Z}\}}(\mathbf{z}|\mathbf{x},t)$ of the state vector, consider a jump of magnitude p_α in the αth component $V_\alpha(t)$ of the generating source process $\{\mathbf{V}(t),\ t\in[0,\infty[\}$. On condition that the system is at the state $\mathbf{Z}(t)=\mathbf{x}$, the increment of the state vector becomes $d\mathbf{Z}(t) = \mathbf{b}_\alpha(\mathbf{x},t)p_\alpha$, where $\mathbf{b}_\alpha(\mathbf{x},t)$ is the αth column of the matrix $\mathbf{b}(\mathbf{x},t)$, cf. (1.105).

Because of the indicated conditioning the state at the time $t+dt$ is deterministic. The transition probability density function can then be represented by the Dirac-delta spike

$$q_{\{\mathbf{Z}\}}(\mathbf{z},t+dt|\mathbf{x},t) = \delta\Big(\mathbf{z} - \big(\mathbf{x} + \mathbf{b}_\alpha(\mathbf{x},t)p_\alpha\big)\Big) \quad (2.54)$$

Since the probability of making a jump into $[p_\alpha,\ p_\alpha + dp_\alpha[$ during the infinitesimal time interval $[t,t+dt[$ is given by $J_{\{V_\alpha\}}(p_\alpha,t)dp_\alpha$, cf. (1.67), the unconditional transition probability is obtained by summing over all contiguous intervals as follows

$$q_{\{\mathbf{Z}\}}(\mathbf{z},t+dt|\mathbf{x},t) = dt\int_{\mathcal{P}_\alpha}\delta\Big(\mathbf{z} - \big(\mathbf{x} + \mathbf{b}_\alpha(\mathbf{x},t)p_\alpha\big)\Big)J_{\{V_\alpha\}}(p_\alpha,t)dp_\alpha \quad (2.55)$$

Relation (2.55) represents the probability density contribution from the jumping from $\mathbf{Z}(t) = \mathbf{x}$ to $\mathbf{Z}(t+dt) = \mathbf{z}$ due to a jump in the αth component. Since the components have been assumed to be mutually statistically independent, the probability density contribution from all l component processes can be obtained as the sum of the contributions, (2.55)

$$q_{\{\mathbf{Z}\}}(\mathbf{z}, t+dt|\mathbf{x}, t) = dt \sum_{\alpha=1}^{l} \int_{\mathcal{P}_\alpha} \delta\Big(\mathbf{z} - \big(\mathbf{x} + \mathbf{b}_\alpha(\mathbf{x}, t)p_\alpha\big)\Big) J_{\{V_\alpha\}}(p_\alpha, t) dp_\alpha \tag{2.56}$$

From (2.10) and (2.56) the jump probability intensity function of the state vector is obtained as

$$J_{\{\mathbf{Z}\}}(\mathbf{z}|\mathbf{x}, t) = \sum_{\alpha=1}^{l} \int_{\mathcal{P}_\alpha} \delta\Big(\mathbf{z} - \big(\mathbf{x} + \mathbf{b}_\alpha(\mathbf{x}, t)p_\alpha\big)\Big) J_{\{V_\alpha\}}(p_\alpha, t) dp_\alpha \tag{2.57}$$

Upon insertion of (2.52), (2.53) and (2.57) into (2.25) and (2.39), respectively, the forward and backward integro-differential Chapman-Kolmogorov operators of the differential system then become

$$\begin{aligned}
&\mathcal{K}_{\mathbf{z},t}\left[q_{\{\mathbf{Z}\}}(\mathbf{z}, t|\mathbf{y}, t_0)\right] = -\sum_{i=1}^{n} \frac{\partial}{\partial z_i}\Big(c_i(\mathbf{z}, t) q_{\{\mathbf{Z}\}}(\mathbf{z}, t|\mathbf{y}, t_0)\Big) + \\
&\frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2}{\partial z_i \partial z_j}\left[\sum_{\alpha=1}^{m} d_{i\alpha}(\mathbf{z}, t) d_{j\alpha}(\mathbf{z}, t) q_{\{\mathbf{Z}\}}(\mathbf{z}, t|\mathbf{y}, t_0)\right] + \\
&\sum_{\alpha=1}^{l} \int_{\mathcal{P}_\alpha} \Big(q_{\{\mathbf{Z}\}}\big(\mathbf{z} - \mathbf{b}_\alpha(t)p_\alpha, t|\mathbf{y}, t_0\big) - q_{\{\mathbf{Z}\}}\big(\mathbf{z}, t|\mathbf{y}, t_0\big)\Big) J_{\{V_\alpha\}}(p_\alpha, t) dp_\alpha
\end{aligned} \tag{2.58}$$

$$\begin{aligned}
&\mathcal{K}^T_{\mathbf{z},t}\left[q_{\{\mathbf{Z}\}}(\mathbf{x}, t_1|\mathbf{z}, t)\right] = \sum_{i=1}^{n} c_i(\mathbf{z}, t) \frac{\partial}{\partial z_i} q_{\{\mathbf{Z}\}}(\mathbf{x}, t_1|\mathbf{z}, t) + \\
&\frac{1}{2}\sum_{i,j=1}^{n}\left[\sum_{\alpha=1}^{m} d_{i\alpha}(\mathbf{z}, t) d_{j\alpha}(\mathbf{z}, t) \frac{\partial^2}{\partial z_i \partial z_j} q_{\{\mathbf{Z}\}}(\mathbf{x}, t_1|\mathbf{z}, t)\right] + \\
&\sum_{\alpha=1}^{l} \int_{\mathcal{P}_\alpha} \Big(q_{\{\mathbf{Z}\}}\big(\mathbf{x}, t_1|\mathbf{z} + \mathbf{b}_\alpha(\mathbf{z}, t)p_\alpha, t\big) - q_{\{\mathbf{Z}\}}\big(\mathbf{x}, t_1|\mathbf{z}, t\big)\Big) J_{\{V_\alpha\}}(p_\alpha, t) dp_\alpha
\end{aligned} \tag{2.59}$$

At the evaluation of (2.58) the vector $\mathbf{b}_\alpha(t)$ has been assumed to be state independent (independent of $\mathbf{x}$). Then the evaluation of the integral of $\delta\Big(\mathbf{z} - \big(\mathbf{x} + \mathbf{b}_\alpha(\mathbf{x}, t)p_\alpha\big)\Big)$ with

respect to $\mathbf{x}$ is very much simplified. If $\mathbf{b}_\alpha = \mathbf{b}_\alpha(\mathbf{x},t)$ i.e. it is state dependent, a preliminary change of variables must be performed, which is defined by the transformations

$$\left.\begin{aligned} &\mathbf{u} = \mathbf{x} + \mathbf{b}_\alpha(\mathbf{x},t)p_\alpha \\ &\mathbf{x} = \mathbf{a}_\alpha(\mathbf{u},p_\alpha,t) \\ &d\mathbf{x} = \frac{d\mathbf{u}}{\left|\det\left(\mathbf{I} + \frac{\partial \mathbf{b}_\alpha}{\partial \mathbf{x}^T}p_\alpha\right)\right|} \end{aligned}\right\} \tag{2.60}$$

where $\mathbf{a}_\alpha(\mathbf{u},p_\alpha,t)$ is the inverse transformation, $\frac{\partial \mathbf{b}_\alpha}{\partial \mathbf{x}^T}$ is the gradient matrix of $\mathbf{b}_\alpha(\mathbf{x},t)$ with respect to $\mathbf{x}$ and $\det\left(\mathbf{I} + \frac{\partial \mathbf{b}_\alpha}{\partial \mathbf{x}^T}p_\alpha\right)$ denotes the Jacobian. Then (2.58) becomes

$$\begin{aligned} &\mathcal{K}_{\mathbf{z},t}\left[q_{\{\mathbf{Z}\}}(\mathbf{z},t|\mathbf{y},t_0)\right] = -\sum_{i=1}^{n}\frac{\partial}{\partial z_i}\left[c_i(\mathbf{z},t)q_{\{\mathbf{Z}\}}(\mathbf{z},t|\mathbf{y},t_0)\right] + \\ &\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2}{\partial z_i \partial z_j}\left[\sum_{\alpha=1}^{m} d_{i\alpha}(\mathbf{z},t)d_{j\alpha}(\mathbf{z},t)q_{\{\mathbf{Z}\}}(\mathbf{z},t|\mathbf{y},t_0)\right] + \\ &\sum_{\alpha=1}^{l}\int_{\mathcal{P}_\alpha}\left[q_{\{\mathbf{Z}\}}\left(\mathbf{a}_\alpha(\mathbf{z},p_\alpha,t),t|\mathbf{y},t_0\right)\frac{1}{\left|\det\left(\mathbf{I} + \frac{\partial \mathbf{b}_\alpha(\mathbf{a}_\alpha(\mathbf{z},p_\alpha,t),t)}{\partial \mathbf{x}^T}p_\alpha\right)\right|} - \right. \\ &\left. q_{\{\mathbf{Z}\}}(\mathbf{z},t|\mathbf{y},t_0)\right] J_{\{V_\alpha\}}(p_\alpha)dp_\alpha \end{aligned} \tag{2.61}$$

Since the jumps of the state vector due to the jumps of the α-th component of the generating source are in the direction of $\mathbf{b}_\alpha(\mathbf{z},t)$ these jumps will only be tangential to the surface $\partial S_t^{(a)}$ if

$$\forall\, t \in]t_0,t_1[\ ,\ \mathbf{z} \in \partial S_t^{(a)}\ :\ \mathbf{n}^T(\mathbf{z},t)\mathbf{b}_\alpha(\mathbf{z},t) = 0 \tag{2.62}$$

2.3 Itô's differential rule and equations for moments

A jump of magnitude p_α in the αth component $V_\alpha(t)$ of the generating source process at the time t results in a jump of magnitude $d\mathbf{Z}(t) = \mathbf{b}_\alpha(\mathbf{Z}(t),t)p_\alpha$ of a state vector $\mathbf{Z}(t)$. Consider now an arbitrary function $f\big(\mathbf{Z}(t),t\big)$ of the state vector $\mathbf{Z}(t)$ and of the time t. A jump of magnitude $d\mathbf{Z}(t) = \mathbf{b}_\alpha(\mathbf{Z}(t),t)p_\alpha$ of the state vector implies a jump of magnitude $df\big(\mathbf{Z}(t),t\big) = \Big(f\Big(\mathbf{Z}(t) + \mathbf{b}_\alpha(\mathbf{Z}(t),t)p_\alpha,t\Big) - f\Big(\mathbf{Z}(t),t\Big)\Big)$ of the function f. The jumps to all contiguous intervals from all l components can then be written as the following sum

$$df\big(\mathbf{Z}(t),t\big) = \sum_{\alpha=1}^{l}\int_{\mathcal{P}_\alpha}\Big(f\Big(\mathbf{Z}(t)+\mathbf{b}_\alpha(\mathbf{Z}(t),t)p_\alpha,t\Big) - f\Big(\mathbf{Z}(t),t\Big)\Big)M_\alpha(dt,t,dp_\alpha,p_\alpha) \tag{2.63}$$

As the startpoint for the derivation of the differential rule we assume the Taylor expansion and we take into account the stochastic equation (1.105).

Making use of the fact that any increment during the infinitesimal time interval is the sum of the increments due to a continuous motion and due to a possible jump, we can write

$$df(\mathbf{Z}(t),t) = f(\mathbf{Z}(t+dt),t+dt) - f(\mathbf{Z}(t),t) = \frac{\partial f(\mathbf{Z}(t),t)}{\partial t}dt +$$

$$\sum_{k=1}^{\infty}\frac{1}{k!}\sum_{i_1,\dots,i_k=1}^{n}\frac{\partial^k f(\mathbf{Z}(t),t)}{\partial Z_{i_1}\cdots\partial Z_{i_k}}\prod_{q=1}^{k}\Big(c_{i_q}(\mathbf{Z}(t),t)dt + \sum_{\alpha_q=1}^{m} d_{i_q\alpha_q}(\mathbf{Z}(t),t)\,dW_{\alpha_q}(t)\Big) +$$

$$\sum_{\alpha=1}^{l}\int_{\mathcal{P}_\alpha}\Big(f\Big(\mathbf{Z}(t)+\mathbf{b}_\alpha(\mathbf{Z}(t),t)p_\alpha,t\Big) - f\Big(\mathbf{Z}(t),t\Big)\Big)M_\alpha(dt,t,dp_\alpha,p_\alpha) =$$

$$\frac{\partial f(\mathbf{Z}(t),t)}{\partial t}dt + \sum_{i=1}^{n}\frac{\partial f(\mathbf{Z}(t),t)}{\partial Z_i}\Big(c_i(\mathbf{Z}(t),t)dt + \sum_{\alpha=1}^{m} d_{i\alpha}(\mathbf{Z}(t),t)\,dW_\alpha(t)\Big) +$$

$$\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2 f(\mathbf{Z}(t),t)}{\partial Z_i\partial Z_j}\sum_{\alpha=1}^{m} d_{i\alpha}(\mathbf{Z}(t),t)\,d_{j\alpha}(\mathbf{Z}(t),t)dt +$$

$$\sum_{\alpha=1}^{l}\int_{\mathcal{P}_\alpha}\Big(f\Big(\mathbf{Z}(t)+\mathbf{b}_\alpha(\mathbf{Z}(t),t)\mathbf{p},t\Big) - f\Big(\mathbf{Z}(t),t\Big)\Big)M_\alpha(dt,t,dp_\alpha,p_\alpha) \tag{2.64}$$

Equation (2.64) is the generalized Itô differential rule for the diffusion and jump excited systems.

Taking the expectation of both sides of (2.64) and using (1.68) the generating equation for moments is obtained as, cf. (2.59)

$$\frac{d}{dt}E\big[f(\mathbf{Z}(t),t)\big] = E\Big[\frac{\partial}{\partial t}f(\mathbf{Z}(t),t)\Big] + E\Big[\mathcal{K}^T_{\mathbf{z},t}\Big(f(\mathbf{Z}(t),t)\Big)\Big] \tag{2.65}$$

where $\mathcal{K}^T_{\mathbf{z},t}[\cdot]$ is the backward integro-differential Chapman-Kolmogorov operator (2.59). Equation (2.65) can alternatively be derived as follows

$$\frac{d}{dt}E\big[f(\mathbf{Z}(t),t)\big] = E\Big[\frac{d}{dt}f(\mathbf{Z}(t),t)\Big] = \int_{S_t}\frac{d}{dt}\Big(f(\mathbf{z},t)q_{\{\mathbf{Z}\}}(\mathbf{z},t\mid\mathbf{y},t_0)\Big)d\mathbf{z} =$$

$$E\Big[\frac{\partial}{\partial t}f(\mathbf{Z}(t),t)\Big] + \int_{S_t} f(\mathbf{z},t)\mathcal{K}_{\mathbf{z},t}\big[q_{\{\mathbf{Z}\}}(\mathbf{z},t\mid\mathbf{y},t_0)\big]d\mathbf{z} =$$

$$E\Big[\frac{\partial}{\partial t}f(\mathbf{Z}(t),t)\Big] + \int_{S_t}\mathcal{K}^T_{\mathbf{z},t}\big[f(\mathbf{z},t)\big]\,q_{\{\mathbf{Z}\}}(\mathbf{z},t\mid\mathbf{y},t_0)d\mathbf{z} =$$

$$E\Big[\frac{\partial}{\partial t}f(\mathbf{Z}(t),t)\Big]+E\Big[\mathcal{K}^T_{\mathbf{z},t}[f(\mathbf{Z}(t),t)]\Big] \tag{2.66}$$

where the commutation of $\mathcal{K}_{\mathbf{z},t}[...]$ and $\mathcal{K}^T_{\mathbf{z},t}[...]$ as specified by (2.49) has been assumed. Hence, it has been implicitly assumed at the derivation of (2.65) and (2.66) that either the entire boundary ∂S_t is non-accessible, i.e. $\partial S_t = \partial S_t^{(2)}$, or the boundary condition (2.40) and the jump condition (2.41) are fulfilled at the accessible boundary $\partial S_t^{(a)}$.

The zero time-lag joint statistical moments of the order N are defined as

$$\mu_{i_1\cdots i_N}(t) = E\Big[Z_{i_1}(t)\cdots Z_{i_N}(t)\Big] \tag{2.67}$$

The corresponding zero time-lag joint central statistical moments of the order N are defined as

$$\kappa_{i_1\cdots i_N}(t) = E\Big[\Big(Z_{i_1}(t)-\mu_{i_1}(t)\Big)\cdots\Big(Z_{i_N}(t)-\mu_{i_N}(t)\Big)\Big] \tag{2.68}$$

Differential equations for these quantities are obtained choosing $f(\mathbf{Z}(t),t) = Z_{i_1}(t)\cdots Z_{i_N}(t)$ and $f(\mathbf{Z}(t),t) = \Big(Z_{i_1}(t)-\mu_{i_1}(t)\Big)\cdots\Big(Z_{i_N}(t)-\mu_{i_N}(t)\Big)$, respectively in (2.65). Hence

$$\frac{d}{dt}\mu_{i_1\cdots i_N}(t) = E\Big[\mathcal{K}^T_{\mathbf{z},t}\Big[Z_{i_1}(t)\cdots Z_{i_N}(t)\Big]\Big] \tag{2.69}$$

$$\begin{aligned}\frac{d}{dt}\kappa_{i_1\cdots i_N}(t) &= E\Big[\frac{\partial}{\partial t}\Big(\big(Z_{i_1}(t)-\mu_{i_1}(t)\big)\cdots\big(Z_{i_N}(t)-\mu_{i_N}(t)\big)\Big)\Big]\\ &+E\Big[\mathcal{K}^T_{\mathbf{z},t}\Big[\Big(Z_{i_1}(t)-\mu_{i_1}(t)\Big)\cdots\Big(Z_{i_N}(t)-\mu_{i_N}(t)\Big)\Big]\Big]\end{aligned} \tag{2.70}$$

These moment equations do not exist for α-stable Lévy motion driven processes. For compound Poisson process driven processes they only exist if the moment $E[P^N]$ of the order N of the marked variable exists.

2.4 Wiener process: Fokker-Planck-Kolmogorov (forward) equation and Kolmogorov backward equation

Consider the system driven by a Wiener process only. The governing stochastic equation (1.105) reduces to the classical Itô's differential equation, [2.2]

$$d\mathbf{Z}(t) = \mathbf{c}\big(\mathbf{Z}(t),t\big)dt + \mathbf{d}\big(\mathbf{Z}(t),t\big)d\mathbf{W}(t) \tag{2.71}$$

The state vector $\mathbf{Z}(t)$ governed by the stochastic equation (2.71) is called a diffusive Markov process.

For this case the names Fokker-Planck-Kolmogorov and Kolmogorov backward operators are coined for the forward and backward differential Chapman-Kolmogorov operators. These become

$$\mathcal{K}_{\mathbf{z},t}\left[q_{\{\mathbf{Z}\}}(\mathbf{z},t|\mathbf{y},t_0)\right] = -\sum_{i=1}^{n}\frac{\partial}{\partial z_i}\Big(c_i(\mathbf{z},t)q_{\{\mathbf{Z}\}}(\mathbf{z},t|\mathbf{y},t_0)\Big) + \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2}{\partial z_i\partial z_j}\Big(D_{ij}(\mathbf{z},t)q_{\{\mathbf{Z}\}}(\mathbf{z},t|\mathbf{y},t_0)\Big) \quad (2.72)$$

$$\mathcal{K}^T_{\mathbf{z},t}\left[q_{\{\mathbf{Z}\}}(\mathbf{x},t_1|\mathbf{z},t)\right] = \sum_{i=1}^{n}c_i(\mathbf{z},t)\frac{\partial}{\partial z_i}q_{\{\mathbf{Z}\}}(\mathbf{x},t_1|\mathbf{z},t) + \frac{1}{2}\sum_{i,j=1}^{n}D_{ij}(\mathbf{z},t)\frac{\partial^2}{\partial z_i\partial z_j}q_{\{\mathbf{Z}\}}(\mathbf{x},t_1|\mathbf{z},t) \quad (2.73)$$

where $D_{ij}(\mathbf{z},t)$ is given by (2.53).

The transition probability density of the diffusive Markov vector process is governed by the so-called Fokker-Planck-Kolmogorov equation, obtained from the forward integro-differential Chapman-Kolmogorov equation (2.24) by omitting the jump part of the equation.

The unconditional probability density function of the state vector becomes

$$f_{\{\mathbf{Z}\}}(\mathbf{z},t) = \int f_{\{\mathbf{Z}\}}(\mathbf{z},t;\mathbf{y},t_0)d\mathbf{y} = \int q_{\{\mathbf{Z}\}}(\mathbf{z},t|\mathbf{y},t_0)f_{\{\mathbf{Z}\}}(\mathbf{y},t_0)d\mathbf{y} \quad (2.74)$$

is also governed by the Fokker-Planck-Kolmogorov equation. The equation (2.24) must be multiplied by $f_{\{\mathbf{Z}\}}(\mathbf{y},t_0)$ and integrated over $\mathbf{y}$, which yields

$$\frac{\partial}{\partial t}f_{\{\mathbf{Z}\}}(\mathbf{z},t) = \mathcal{K}_{\mathbf{z},t}\left[f_{\{\mathbf{Z}\}}(\mathbf{z},t)\right] = -\sum_{i}\frac{\partial}{\partial z_i}\Big(c_i(\mathbf{z},t)f_{\{\mathbf{Z}\}}(\mathbf{z},t)\Big) + \frac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial z_i\partial z_j}\Big(D_{ij}(\mathbf{z},t)f_{\{\mathbf{Z}\}}(\mathbf{z},t)\Big) \quad (2.75)$$

The relevant initial condition is then obtained from (2.75) as

$$f_{\mathbf{Z}}(\mathbf{z},t)\big|_{t=t_0} = f_{\mathbf{Z}}(\mathbf{z},t_0) \quad (2.76)$$

The differential rule (2.64) reduces to the classical Itô's differential rule, [2.2]

$$df\big(\mathbf{Z}(t),t\big) = \frac{\partial f(\mathbf{Z}(t),t)}{\partial t}dt + \sum_{i=1}^{n}\frac{\partial f(\mathbf{Z}(t),t)}{\partial Z_i}\Big(c_i\big(\mathbf{Z}(t),t\big)dt + \sum_{\alpha=1}^{m}d_{i\alpha}\big(\mathbf{Z}(t),t\big)dW_\alpha(t)\Big) + \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2 f(\mathbf{Z}(t),t)}{\partial Z_i\partial Z_j}\sum_{\alpha=1}^{m}d_{i\alpha}\big(\mathbf{Z}(t),t\big)d_{j\alpha}\big(\mathbf{Z}(t),t\big)dt \quad (2.77)$$

2.5 Compound Poisson process: Kolmogorov-Feller forward and backward equations

Consider now the system driven by a compound Poisson process, i.e. by a Poisson distributed train of impulses, only. The governing stochastic equation (1.105) reduces to the following differential equation

$$d\mathbf{Z}(t) = \mathbf{c}\big(\mathbf{Z}(t),t\big)dt + \mathbf{b}\big(\mathbf{Z}(t),t\big)d\mathbf{V}(t) \tag{2.78}$$

where $\mathbf{V}(t)$ represents the multivariate compound Poisson process, hence its components are given by

$$dV_\alpha(t) = \int\limits_{\mathcal{P}_\alpha} p_\alpha M_\alpha(dt,t,dp_\alpha,p_\alpha) \tag{2.79}$$

or, equivalently

$$dV_\alpha(t) = P_\alpha(t)dN_\alpha(t) \tag{2.80}$$

The state vector $\mathbf{Z}(t)$ governed by the stochastic equation (2.78) is a non-diffusive so-called Poisson driven Markov process.

The jump probability intensity function, in the case of each component compound Poisson process, is given as, cf. (1.69)

$$J_{\{V_\alpha\}}(p_\alpha,t) = \nu_\alpha(t) f_{P_\alpha}(p_\alpha) \tag{2.81}$$

hence the jump probability intensity function of the dynamic system becomes, cf. (2.57)

$$J_{\{\mathbf{Z}\}}\big(\mathbf{z} \mid \mathbf{x},t\big) = \sum_{\alpha=1}^{l} \nu_\alpha(t) \int\limits_{\mathcal{P}_\alpha} \delta\Big(\mathbf{z} - \big(\mathbf{x} + \mathbf{b}_\alpha(\mathbf{x},t)p_\alpha\big)\Big) f_{P_\alpha}(p_\alpha)dp_\alpha \tag{2.82}$$

The forward integro-differential Chapman-Kolmogorov operator corresponding to (2.58) becomes

$$\mathcal{K}_{\mathbf{z},t}\left[q_{\{\mathbf{Z}\}}(\mathbf{z},t|\mathbf{y},t_0)\right] = -\sum_i \frac{\partial}{\partial z_i}\Big(c_i(\mathbf{z},t)q_{\{\mathbf{Z}\}}(\mathbf{z},t|\mathbf{y},t_0)\Big) +$$

$$\sum_{\alpha=1}^{l} \nu_\alpha(t) \int\limits_{\mathcal{P}_\alpha} \Big(q_{\{\mathbf{Z}\}}(\mathbf{z} - \mathbf{b}_\alpha(t)p_\alpha,t|\mathbf{y},t_0) - q_{\{\mathbf{Z}\}}(\mathbf{z},t|\mathbf{y},t_0)\Big) f_{P_\alpha}(p_\alpha)dp_\alpha \tag{2.83}$$

In a more general case of a state dependent matrix $\mathbf{b}_\alpha = \mathbf{b}_\alpha(\mathbf{x},t)$ the forward integro-differential Chapman-Kolmogorov operator is, cf. (2.61)

$$\mathcal{K}_{\mathbf{z},t}\left[q_{\{\mathbf{Z}\}}(\mathbf{z},t|\mathbf{y},t_0)\right] = -\sum_i \frac{\partial}{\partial z_i}\Big(c_i(\mathbf{z},t)q_{\{\mathbf{Z}\}}(\mathbf{z},t|\mathbf{y},t_0)\Big) +$$
$$\sum_{\alpha=1}^{l}\nu_\alpha(t)\int_{\mathcal{P}_\alpha}\Bigg(q_{\{\mathbf{Z}\}}\big(\mathbf{a}_\alpha(\mathbf{z},p_\alpha,t),t|\mathbf{y},t_0\big)\frac{1}{\left|\det\left(\mathbf{I}+\frac{\partial \mathbf{b}_\alpha(\mathbf{a}_\alpha(\mathbf{z},p_\alpha,t),t)}{\partial \mathbf{x}^T}p_\alpha\right)\right|} -$$
$$q_{\{\mathbf{Z}\}}\big(\mathbf{z},t|\mathbf{y},t_0\big)\Bigg)f_{P_\alpha}(p_\alpha)dp_\alpha \tag{2.84}$$

The backward integro-differential Chapman-Kolmogorov operator corresponding to (2.59) becomes

$$\mathcal{K}^T_{\mathbf{z},t}\left[q_{\{\mathbf{Z}\}}(\mathbf{x},t_1|\mathbf{z},t)\right] = \sum_{i=1}^{n} c_i(\mathbf{z},t)\frac{\partial}{\partial z_i}q_{\{\mathbf{Z}\}}(\mathbf{x},t_1|\mathbf{z},t) -$$
$$\sum_{\alpha=1}^{s}\nu_\alpha(t)\int_{\mathcal{P}_\alpha}\Big(q_{\{\mathbf{Z}\}}\big(\mathbf{x},t_1|\mathbf{z}+\mathbf{b}_\alpha(\mathbf{z},t)p_\alpha,t\big) - q_{\{\mathbf{Z}\}}\big(\mathbf{x},t_1|\mathbf{z},t\big)\Big)f_{P_\alpha}(p_\alpha)dp_\alpha \tag{2.85}$$

In the case of the equation governing the usual, unconditional probability density $f_{\mathbf{Z}}(\mathbf{z},t)$ the forward integro-differential Chapman-Kolmogorov operator, has the same form as (2.83) or (2.84). For example in the case of the state independent matrix $\mathbf{b}_\alpha(t)$ this operator is

$$\mathcal{K}_{\mathbf{z},t}\left[f_{\{\mathbf{Z}\}}(\mathbf{z},t)\right] = -\sum_i \frac{\partial}{\partial z_i}\Big(c_i(\mathbf{z},t)f_{\{\mathbf{Z}\}}(\mathbf{z},t)\Big) +$$
$$\sum_{\alpha=1}^{l}\nu_\alpha(t)\int_{\mathcal{P}_\alpha}\Big(f_{\{\mathbf{Z}\}}\big(\mathbf{z}-\mathbf{b}_\alpha(t)p_\alpha,t\big) - f_{\{\mathbf{Z}\}}(\mathbf{z},t)\Big)f_{P_\alpha}(p_\alpha)dp_\alpha \tag{2.86}$$

Alternatively, using a Taylor expansion of the function $f_{\{\mathbf{Z}\}}(\mathbf{z}-\mathbf{b}_\alpha(t)p_\alpha,t)$, the forward integro-differential Chapman-Kolmogorov operator (2.86) is recast into

$$\mathcal{K}_{\mathbf{z},t}\left[f_{\{\mathbf{Z}\}}(\mathbf{z},t)\right] = -\sum_i \frac{\partial}{\partial z_i}\Big(c_i(\mathbf{z},t)f_{\{\mathbf{Z}\}}(\mathbf{z},t)\Big) +$$
$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\sum_{\alpha=1}^{l}\nu_\alpha(t)E[P_\alpha^n]\frac{\partial^n}{\partial z_{i_1}\cdots\partial z_{i_n}}\Big(f_{\{\mathbf{Z}\}}(\mathbf{z},t)b_{i_1\alpha}\big(\mathbf{z}(t),t\big)\cdots b_{i_n\alpha}\big(\mathbf{z}(t),t\big)\Big) \tag{2.87}$$

(2.87) is known as the Kramer-Moyal expansion of the forward integro-differential Chapman-Kolmogorov resulting in a purely differential formulation. The differential rule reduces (2.64) to the following integro-differential form

$$df\big(\mathbf{Z}(t),t\big) = \frac{\partial f\big(\mathbf{Z}(t),t\big)}{\partial t}dt + \sum_{i=1}^{n}\frac{\partial f\big(\mathbf{Z}(t),t\big)}{\partial Z_i}c_i\big(\mathbf{Z}(t),t\big)dt +$$

$$\sum_{\alpha=1}^{l} \int_{\mathcal{P}_\alpha} \Big(f\Big(\mathbf{Z}(t) + \mathbf{b}_\alpha(\mathbf{Z}(t),t)p_\alpha, t\Big) - f\Big(\mathbf{Z}(t),t\Big)\Big) M_\alpha(dt,t,dp_\alpha,p_\alpha) \tag{2.88}$$

or to the equivalent form

$$df(\mathbf{Z}(t),t) = \frac{\partial f(\mathbf{Z}(t),t)}{\partial t}dt + \sum_{i=1}^{n} \frac{\partial f(\mathbf{Z}(t),t)}{\partial Z_i} c_i(\mathbf{Z}(t),t)\,dt +$$
$$\sum_{\alpha=1}^{l} \Big(f\Big(\mathbf{Z}(t) + \mathbf{b}_\alpha(\mathbf{Z}(t),t)P_\alpha(t),\ t\Big) - f\Big(\mathbf{Z}(t),\ t\Big)\Big) dN_\alpha(t) \tag{2.89}$$

Upon a Taylor expansion of the function $f\Big(\mathbf{Z}(t)+\mathbf{b}_\alpha(\mathbf{Z}(t),t)p_\alpha, t\Big)$ the following Kramer-Moyal expansion (2.88) is obtained as

$$df(\mathbf{Z}(t),t) = \frac{\partial f(\mathbf{Z}(t),t)}{\partial t}dt + \sum_{i=1}^{n} \frac{\partial f(\mathbf{Z}(t),t)}{\partial Z_i} c_i(\mathbf{Z}(t),t)\,dt$$
$$+ \sum_{\alpha=1}^{l}\sum_{n=1}^{\infty} \frac{1}{n!} \frac{\partial^n f(\mathbf{Z},t)}{\partial Z_{i_1}\cdots\partial Z_{i_n}} b_{i_1\alpha}(\mathbf{Z}(t),t)\cdots b_{i_n\alpha}(\mathbf{Z}(t),t) \int_{\mathcal{P}_\alpha} p_\alpha^n M_\alpha(dt,t,dp_\alpha,p_\alpha) \tag{2.90}$$

2.6 α-stable Lévy motion

Stochastic equation governing the behaviour of the system driven by α-stable Lévy motion is of the form

$$d\mathbf{Z}(t) = \mathbf{c}(\mathbf{Z}(t),t)dt + \mathbf{b}(\mathbf{Z}(t),t)d\mathbf{V}(t) \tag{2.91}$$

where $\{\mathbf{V}(t),\ t \in [0,\infty[\}$ represents a vector of statistically independent components of α-stable Lévy motions $\{V_\alpha(t),\ t \in [0,\infty[\}$.

The jump probability intensity function $J_{\{\mathbf{Z}\}}(\mathbf{z}|\mathbf{x},t)$ of the state vector is then given by (2.57), where the jump probability intensity function $J_{\{V_\alpha\}}(p_\alpha,t)$ of the α-th component process is given by (1.71) and (1.73).

Since α-stable Lévy motion is a process with independent increments the state vector $\mathbf{Z}(t)$ is a Markov process, driven by an α-stable Lévy motion.

The forward integro-differential Chapman-Kolmogorov operator corresponding to (2.58) becomes

$$\mathcal{K}_{\mathbf{z},t}\left[q_{\{\mathbf{Z}\}}(\mathbf{z},t|\mathbf{y},t_0)\right] = -\sum_{i=1}^{n} \frac{\partial}{\partial z_i}\Big(c_i(\mathbf{z},t)q_{\{\mathbf{Z}\}}(\mathbf{z},t|\mathbf{y},t_0)\Big) +$$

$$\sum_{\alpha=1}^{l} \int_{\mathcal{P}_\alpha} \Big(q_{\{\mathbf{Z}\}}\left(\mathbf{z} - \mathbf{b}_\alpha(t)p_\alpha, t|\mathbf{y}, t_0\right) - q_{\{\mathbf{Z}\}}\left(\mathbf{z}, t|\mathbf{y}, t_0\right) \Big) J_{\{V_\alpha\}}(p_\alpha, t) dp_\alpha \tag{2.92}$$

The backward integro-differential Chapman-Kolmogorov operator corresponding to (2.59) becomes

$$\mathcal{K}^T_{\mathbf{z},t}\left[q_{\{\mathbf{Z}\}}(\mathbf{x}, t_1|\mathbf{z}, t)\right] = \sum_{i=1}^{n} c_i(\mathbf{z}, t) \frac{\partial}{\partial z_i} q_{\{\mathbf{Z}\}}(\mathbf{x}, t_1|\mathbf{z}, t) -$$

$$\sum_{\alpha=1}^{l} \int_{\mathcal{P}_\alpha} \Big(q_{\{\mathbf{Z}\}}\left(\mathbf{x}, t_1|\mathbf{z} + \mathbf{b}_\alpha(\mathbf{z}, t)p_\alpha, t\right) - q_{\{\mathbf{Z}\}}\left(\mathbf{x}, t_1|\mathbf{z}, t\right) \Big) J_{\{V_\alpha\}}(p_\alpha, t) dp_\alpha \tag{2.93}$$

2.7 Bibliography

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2.8 Example problems

1.1 Verify the condition of the continuity of the sample paths and evaluate the jump intensity function for the α-stable Lévy motion with Lévy (sometimes called Pearson) distribution, i.e. $S_\alpha\left((\Delta t)^{1/\alpha}, \beta, 0\right)$, for $\alpha = 1/2, \beta = 1$. The general form of the density function of the Lévy distribution is

$$f_{1/2}(x) = \left(\frac{\sigma}{2\pi}\right)^{1/2} (x-\mu)^{-3/2} \exp\left\{-\frac{\sigma}{2(x-\mu)}\right\}$$

1.2 Formulate the Master equation for the random telegraph process (wave) (the process which has a discrete state space))

$$X(t) = (-1)^{N(t)}$$

where $N(t)$ is the homogeneous Poisson process with the mean rate $\nu = const.$

1.3 Using the usual Itô's differential rule derive the equations for first and second order moments for a linear SDOF system driven by a Wiener process. Evaluate stationary mean value and variance.

1.4 Using the generalized Itô's differential rule derive the equations for first and second order moments for a linear SDOF system driven by a compound Poisson process. Evaluate stationary mean value and variance.

CHAPTER 3

DYNAMIC RESPONSE OF NON-LINEAR SYSTEMS TO GAUSSIAN WHITE NOISE AND FILTERED GAUSSIAN WHITE NOISE EXCITATIONS. DIFFUSIVE MARKOV PROCESS TECHNIQUES

3.1 Review of available analytical solutions of the Fokker-Planck-Kolmogorov equation

Before we proceed to formulate the solutions techniques to the F-P-K equation it should be clearly stated that the attention will be confined to the dynamical systems which can reach a steady, or stationary, state, e.g. asymptotically stable dynamical systems.

3.1.1 Solution in the univariate case

The simplest vibratory system, i.e. a single-degree-of-freedom system, is represented by a two-dimensional state vector (1.90), (1.91). Here we will start, however with a univariate problem, or a one-dimensional state vector. This is not merely for the sake of pure formalism. For example with the help of the stochastic averaging technique, the problem of the stochastic response of an oscillator can be converted to two uncoupled stochastic differential equations, one of which governs the random amplitude process, and the other the random phase process.

Let us consider the univariate process, or one dimensional state vector, $\mathbf{Z}(t) = z(t)$. The Fokker-Planck-Kolmogorov equation becomes, cf. (2.75)

$$\frac{\partial}{\partial t} f_{\{Z\}}(z,t) = -\frac{\partial}{\partial z}\Big(c(z,t) f_{\{Z\}}(z,t)\Big) + \frac{1}{2}\frac{\partial^2}{\partial z^2}\Big(D(z,t) f_{\{Z\}}(z,t)\Big) \tag{3.1}$$

If we introduce the so-called probability current

$$G(z,t) = c(z,t) f_{\{Z\}}(z,t) - \frac{1}{2}\frac{\partial}{\partial z}\Big(D(z,t) f_{\{Z\}}(z,t)\Big) \tag{3.2}$$

we see that it satisfies the following, equivalent to (3.1), equation

$$\frac{\partial}{\partial t} f_{\{Z\}}(z,t) + \frac{\partial}{\partial z} G(z,t) = 0 \tag{3.3}$$

which expresses the condition of conservation of probability.

If the state variable $Z(t) \in]-\infty, \infty[$, i.e. it can assume values from the entire real line, the F-P-K equation is valid on the entire real line and the boundary conditions must be given for $-\infty$ and $+\infty$. Upon integrating the equation (3.3) over z and using the normalization condition

$$\int_{-\infty}^{\infty} f_{\{Z\}}(z,t)\,dz = 1 \tag{3.4}$$

which must hold for any time t, we obtain the conditions

$$G(-\infty,t) = G(\infty,t) \tag{3.5}$$

Usually, however one of the following conditions are satisfied at $z = \pm\infty$

$$f_{\{Z\}}(-\infty,t) = f_{\{Z\}}(\infty,t) = 0 \tag{3.6}$$

$$G(-\infty,t) = G(\infty,t) = 0 \tag{3.7}$$

The initial condition for the probability density at the time $t = 0$ can be assumed as

$$f_{\{Z\}}(z,0) = f^0_{\{Z\}}(z) \tag{3.8}$$

The probability density $f_{\{Z\}}(z,t)$ as a solution of the differential equation is uniquely determined by the initial and boundary conditions.

Stationary solution

If the drift and diffusion coefficients are time independent, and the dynamical system is asymptotically stable, as $t \to \infty$ the probability distribution approaches a stationary one, with the stationary density function $f^s_{\{Z\}}(z)$ which is independent of time and of the initial density $f^0_{\{Z\}}(z)$. Hence the F-P-K equation becomes homogeneous

$$-\frac{\partial}{\partial z}\Big(c(z)f^s_{\{Z\}}(z)\Big) + \frac{1}{2}\frac{\partial^2}{\partial z^2}\Big(D(z)f^s_{\{Z\}}(z)\Big) = \frac{\partial}{\partial t}f^s_{\{Z\}}(z) = 0 \tag{3.9}$$

The fact that $\partial f^s_{\{Z\}}(z)/\partial t = 0$ implies also that

$$G_s(z) = \text{const} = G_s \tag{3.10}$$

Integration of equation (3.9) yields

$$-c(z)f^s_{\{Z\}}(z) + \frac{1}{2}\frac{\partial}{\partial z}\Big(D(z)f^s_{\{Z\}}(z)\Big) = \text{const} = G_s \tag{3.11}$$

For $v(z) = D(z)f^s_{\{Z\}}(z)$ we obtain the differential equation

$$\frac{d}{dz}v(z) - 2\frac{c(z)}{D(z)}v(z) = 2G_s \tag{3.12}$$

The solution of this first-order, ordinary non-homogeneous differential equation, subject to zero boundary conditions (3.7) is found, by means of standard methods to be

$$f^s_{\{Z\}}(z) = \frac{C}{D(z)}\exp\left(2\int_0^z \frac{c(y)}{D(y)}dy\right) \tag{3.13}$$

The constant C has to be determined from the normalization condition (3.4). Since $f_{\{Z\}}(-\infty) = f_{\{Z\}}(\infty)$ and $D(y) > 0$ a stationary solution does only exist for $c(z) < 0$ for $z > 0$ and $c(z) > 0$ for $z < 0$.

In the case of non-zero constant probability current, assuming that the process takes place on an interval $[a, b]$. Then we may impose the boundary conditions

$$f^s_{\{Z\}}(a) = f^s_{\{Z\}}(b) \tag{3.14}$$

$$G_s(a) = G_s(b) \tag{3.15}$$

The solution of the equation (3.12) becomes

$$f^s_{\{Z\}}(z) = \frac{f^s_{\{Z\}}(a)}{\frac{D(z)}{\psi(z)} \int_a^b \frac{dy}{\psi(y)}} \left(\frac{D(b)}{\psi(b)} \int_a^x \frac{dy}{\psi(y)} + \frac{D(a)}{\psi(a)} \int_x^b \frac{dy}{\psi(y)} \right) \tag{3.16}$$

where

$$\psi(y) = \exp\left(2 \int_a^x \frac{c(y)dy}{D(y)} \right) \tag{3.17}$$

3.1.2 Solution in the bivariate case – non-linear oscillator

Consider now a non-linear oscillator under a non-zero mean Gaussian white noise excitation with spectral density S_0, governed by the equation of motion

$$\ddot{Y} + 2\zeta\omega_0\dot{Y} + f(Y) = \dot{W}(t) \tag{3.18}$$

Equivalently, the state vector $\mathbf{Z} = [Z_1, Z_2]^T = [Y, \dot{Y}]^T$ is governed by the stochastic equations

$$\left.\begin{aligned} dZ_1 &= Z_2 dt \\ dZ_2 &= \big(-2\zeta\omega_0 Z_2 + f(Z_1)\big)dt + dW(t) \end{aligned}\right\} \tag{3.19}$$

The derivate moments are

$$\left.\begin{aligned} c_1(\mathbf{z}, t) = z_2 \qquad & c_2(\mathbf{z}, t) = -2\zeta\omega_0 z_2 - f(z_1) \\ D_{11} = D_{12} = D_{21} = 0 \quad , \quad & D_{22} = 2\pi S_0 \end{aligned}\right\} \tag{3.20}$$

The Fokker-Planck-Kolmogorov equation

$$\frac{\partial}{\partial t} f_{\{\mathbf{Z}\}}(\mathbf{z},t) = -\sum_{i}^{2} \frac{\partial}{\partial z_i}\Big(c_i(\mathbf{z},t) f_{\{\mathbf{Z}\}}(\mathbf{z},t)\Big) + \frac{1}{2}\sum_{i,j}^{2} \frac{\partial^2}{\partial z_i \partial z_j}\Big(D_{ij}(\mathbf{z},t) f_{\{\mathbf{Z}\}}(\mathbf{z},t)\Big) \quad (3.21)$$

specified for the stationary case becomes

$$0 = -z_2 \frac{\partial f_{\{\mathbf{Z}\}}(z_1,z_2)}{\partial z_1} + (2\zeta\omega_0 z_2 + f(z_1)) \frac{\partial f_{\{\mathbf{Z}\}}(z_1,z_2)}{\partial z_2} + 2\zeta\omega_0 f_{\{\mathbf{Z}\}}(z_1,z_2) +$$

$$\pi S_0 \frac{\partial^2}{\partial z_2^2} f_{\{\mathbf{Z}\}}(z_1,z_2) \quad (3.22)$$

If $f(Z)$ is an odd function i.e. $f(-Z) = -f(Z)$, and since the excitation is zero-mean in the stationary state, it follows

$$E[Z_1(t)] = E[Z_2(t)] = 0 \quad (3.23)$$

Moreover, it can be shown (see Example Problem 3.1) that

$$E[Z_1(t)Z_2(t)] = 0 \quad (3.24)$$

Hence the stationary displacement $Z_1(t)$ and velocity response $Z_2(t)$ processes are uncorrelated.

The fact that the two processes are uncorrelated does not necessarily imply that they are independent. Nevertheless we will try the solution in the product form

$$f_{\{\mathbf{Z}\}}(z_1,z_2) = f_{\{Z_1\}}(z_1) f_{\{Z_2\}}(z_2) \quad (3.25)$$

Substitution of (3.25) into equation (3.22) and division by $f_{\{Z_1\}}(z_1) f_{\{Z_2\}}(z_2)$ yields

$$0 = -\frac{z_2}{f_{\{Z_1\}}} \frac{df_{\{Z_1\}}(z_1)}{dz_1} + (2\zeta\omega_0 z_2 + f(z_1)) \frac{1}{f_{\{Z_2\}}} \frac{df_{\{Z_2\}}(z_2)}{dz_2} + 2\zeta\omega_0 +$$

$$\frac{\pi S_0}{f_{\{Z_2\}}(z_2)} \frac{d^2 f_{\{Z_2\}}(z_2)}{dz_2^2} \quad (3.26)$$

It follows that the solution in the product form (3.25) is admitted if the following equation is satisfied

$$\frac{z_2}{f_{\{Z_1\}}} \frac{df_{\{Z_1\}}(z_1)}{dz_1} = f(z_1) \frac{df_{\{Z_2\}}(z_2)}{dz_2} \quad (3.27)$$

and the remaining part of (3.25) is fulfilled by $f_{\{Z\}}(z_2)$. (3.26) can be rewritten in the form of

$$\frac{1}{f(z_1)}\frac{df_{\{Z_1\}}(z_1)}{dz_1} = \frac{1}{z_2}\frac{df_{\{Z_2\}}(z_2)}{dz_2} = \text{const} = A \tag{3.28}$$

where the standard argumentation of the separation method has been used. Integrating (3.28) we obtain

$$\left.\begin{aligned} f_{\{Z_1\}}(z_1) &= C_1 \exp\left(-A\int_0^{z_1} f(u)du\right) \\ f_{\{Z_2\}}(z_2) &= C_2 \exp\left(-\frac{A}{2}z_2^2\right) \end{aligned}\right\} \tag{3.29}$$

Substituting $f_{\{Z_2\}}(z_2)$ given by (3.29) into the part of (3.26) not considered by (3.29) we obtain a quadratic equation for A. One root depends on z_2, but the other is constant

$$A = \frac{2\zeta\omega_0}{\pi S_0} \tag{3.30}$$

Consequently the joint probability density of the stationary displacement $Z_1(t)$ and velocity response $Z_2(t)$ processes is given by

$$f_{\{\mathbf{Z}\}}(z_1, z_2) = C\exp\left(-\frac{2\zeta\omega_0}{\pi S_0}\left[\frac{z_2^2}{2} + U(z_1)\right]\right) \tag{3.31}$$

where $U(z_1) = \int_0^{z_1} f(u)du)$ is the strain energy of the system. The constant C is evaluated from the normalization condition

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{\{\mathbf{Z}\}}(z_1, z_2)dz_1 dz_2 = 1 \tag{3.32}$$

The obtained result needs comments. The displacement and velocity responses are independent. The one-dimensional probability density $f_{\{Z_2\}}(z_2, t)$ of the velocity response process $Z_2(t)$ is Gaussian. However a random process can only be Gaussian if all its finite-dimensional distributions are Gaussian. This is not the case here, because the mean-square integral of the velocity response process, i.e. the displacement response process is not Gaussian. In other words the values of the velocity process at different times are not jointly Gaussian distributed and the velocity process is not Gaussian.

In passing we note that the sum of the terms in the exponent of (3.31) is just sum of the kinetic and of the strain energy of the system, i.e. the total mechanincal energy

$$E(z_1, z_2) = \frac{z_2^2}{2} + U(z_1) \tag{3.33}$$

Hence the equation (3.31) can be written as

$$f_{\{\mathbf{Z}\}}(z_1, z_2) = C \exp\left(-\frac{2\zeta\omega_0}{\pi S_0} E(z_1, z_2)\right) \tag{3.34}$$

The analytical solution is available in a more general case, when the non-linear function is of the form

$$g(z_1, z_2) = 2\zeta\omega_0 z_2 w(E) + f(z_1) \tag{3.35}$$

where $w(\cdot)$ is a positive monotonically increasing function. The probability density function is then written as, Caughey [3.8]

$$f_{\{\mathbf{Z}\}}(z_1, z_2) = C \exp\left(-\frac{2\zeta\omega_0}{\pi S_0} \int_0^{E(z_1,z_2)} w(u)du\right) \tag{3.36}$$

3.1.3 Solution in the multivariate case – MDOF non-linear systems

The stationary solution of the Fokker-Planck-Kolmogorov equation for a MDOF non-linear system can be obtained under the following conditions:

- white noise excitation processes are uncorrelated,
- both the generalized inertia and the generalized damping forces must be uncoupled,
- the ratio of every generalized damping coefficient c_j to the corresponding generalized excitation spectral density S_{jj}^0 must be constant.

Then the stationary solution of the Fokker-Planck-Kolmogorov equation is written as, Caughey [3.8]

$$f_{\{\mathbf{Z}\}}(\mathbf{z}) = C \exp\left(-\frac{\gamma}{\pi}\left[\frac{1}{2}\sum_j m_j z_{2j}^2 + U(\mathbf{z})\right]\right) \tag{3.37}$$

where z_{2j} denotes the variable corresponding with the velocity process.

3.2 Closure approximations for hierarchy of moment equations

3.2.1 Differential equations for moments

It is expedient to formulate the equations for zero-mean response processes, or centralized state variables.

Equations for mean values are

$$\frac{d}{dt}\mu_i(t) = E\Big[c_i\big(\mathbf{Z}(t),t\big)\Big] \tag{3.38}$$

Equations for centralized state variables $Z_i^0(t) = Z_i(t) - \mu_i(t)$ become, cf. (2.71)

$$d\mathbf{Z}^0(t) = \mathbf{c}^0\big(\mathbf{Z}(t),t\big)dt + \mathbf{d}\big(\mathbf{Z}(t),t\big)d\mathbf{W}(t) \tag{3.39}$$

where the centralized drift vector is

$$\mathbf{c}^0\big(\mathbf{Z}(t),t\big) = \mathbf{c}\big(\mathbf{Z}(t),t\big) - E\Big[\mathbf{c}\big(\mathbf{Z}(t),t\big)\Big] \tag{3.40}$$

Joint central moments of order s of the state vector are defined as

$$\kappa_{i_1 i_2 \cdots i_s}(t) = E\left[\prod_{k=1}^{s}\Big(Z_{i_k}(t) - \mu_{i_k}(t)\Big)\right] \tag{3.41}$$

Using (2.66), (2.73) with $f\big(\mathbf{Z}(t),t\big) = \prod_{k=1}^{s}\big(Z_{i_k}(t) - \mu_{i_k}(t)\big)$ the following differential equations for the joint central moments may be derived

$$\left.\begin{aligned}
&\frac{d}{dt}\kappa_{ij}(t) = 2\left\{E\big[Z_i^0 c_j^0\big(\mathbf{Z}(t),t\big)\big]\right\}_s + \sum_{\alpha=1}^{m}\left\{E\big[d_{i\alpha}\big(\mathbf{Z}(t),t\big)d_{j\alpha}\big(\mathbf{Z}(t),t\big)\big]\right\}_s \\
&\frac{d}{dt}\kappa_{ijk}(t) = 3\left\{E\big[Z_i^0 Z_j^0 c_k^0\big(\mathbf{Z}(t),t\big)\big]\right\}_s + 3\sum_{\alpha=1}^{m}\left\{E\big[d_{i\alpha}\big(\mathbf{Z}(t),t\big)d_{j\alpha}\big(\mathbf{Z}(t),t\big)Z_k^0\big]\right\}_s \\
&\frac{d}{dt}\kappa_{ijkl}(t) = 4\left\{E\left[Z_i^0 Z_j^0 Z_k^0 c_l^0\big(\mathbf{Z}(t),t\big)\right]\right\}_s + \\
&6\sum_{\alpha=1}^{m}\left\{E\big[d_{i\alpha}\big(\mathbf{Z}(t),t\big)d_{j\alpha}\big(\mathbf{Z}(t),t\big)Z_k^0 Z_l^0\big]\right\}_s \\
&\vdots \\
&\frac{d}{dt}\kappa_{i_1\cdots i_s}(t) = s\left\{E\left[Z_{i_1}^0\cdots Z_{i_{s-1}}^0 c_{i_s}^0\big(\mathbf{Z}(t),t\big)\right]\right\}_s + \\
&\frac{s(s-1)}{2}\sum_{\alpha=1}^{m}\left\{E[d_{i_1\alpha}\big(\mathbf{Z}(t),t\big)d_{i_2\alpha}\big(\mathbf{Z}(t),t\big)Z_{i_3}^0\cdots Z_{i_s}^0]\right\}_s
\end{aligned}\right\} \tag{3.42}$$

$\{\cdot\}_s$ signifies the so-called Stratonovich permulation symbol, which takes the mean of all permutations of the indicated free indices. As an example $\{Z_i^0 Z_j^0 C_k^0\}_s = \frac{1}{3}\big(Z_i^0 Z_j^0 C_k^0 + Z_j^0 Z_k^0 C_j^0 + Z_k^0 Z_i^0 C_j^0\big)$. In the case of state-independent diffusion terms the equations for moments are

$$
\left.
\begin{aligned}
&\frac{d}{dt}\kappa_{ij}(t) = 2\left\{E\left[Z_i^0 c_j^0(\mathbf{Z}(t),t)\right]\right\}_s + \sum_{\alpha=1}^{m}\left\{d_{i\alpha}d_{j\alpha}\right\}_s \\
&\frac{d}{dt}\kappa_{ijk}(t) = 3\left\{E\left[Z_i^0 Z_j^0 c_k^0(\mathbf{Z}(t),t)\right]\right\}_s \\
&\frac{d}{dt}\kappa_{ijkl}(t) = 4\left\{E\left[Z_i^0 Z_j^0 Z_k^0 c_l^0(\mathbf{Z}(t),t)\right]\right\}_s + 6\sum_{\alpha=1}^{m}\left\{d_{i\alpha}d_{j\alpha}\kappa_{kl}\right\}_s \\
&\vdots \\
&\frac{d}{dt}\kappa_{i_1\cdots i_s}(t) = s\left\{\left[Z_{i_1}^0\cdots Z_{i_{s-1}}^0 c_{i_s}^0(\mathbf{Z}(t),t)\right]\right\}_s + \\
&\frac{s(s-1)}{2}\sum_{\alpha=1}^{m}\left\{d_{i_1\alpha}d_{i_2\alpha}\kappa_{i_3\cdots i_s}\right\}_s
\end{aligned}
\right\} \tag{3.43}
$$

Of special interest is the case of a system with polynomial non-linearity, for example when it is given as a cubic form of the state variables. In this case the centralized drift term is written as

$$
c_i^0(\mathbf{Z}(t),t) = A_i + B_{im}Z_m^0 + C_{imn}Z_m^0 Z_n^0 + D_{imnp}Z_m^0 Z_n^0 Z_p^0 \tag{3.44}
$$

Consequently the expectations of the drift term multiplied by the state variables can be explicitly performed to give the following expressions for the central moments and the equations for moments

$$
\left.
\begin{aligned}
&\frac{d}{dt}\kappa_{ij}(t) = 2\left\{B_{im}\kappa_{mj}\right\}_s + 2\left\{C_{imn}\kappa_{mnj}\right\}_s + 2\left\{D_{imnp}\kappa_{mnpj}\right\}_s + \\
&\sum_{\alpha=1}^{m}\left\{d_{i\alpha}d_{j\alpha}\right\}_s \\
&\frac{d}{dt}\kappa_{ijk}(t) = 3\left\{A_i\kappa_{jk}\right\}_s + 3\left\{B_{im}\kappa_{mjk}\right\}_s + 3\left\{C_{imn}\kappa_{mnjk}\right\}_s + \\
&3\left\{D_{imnp}\kappa_{mnpjk}\right\}_s \\
&\frac{d}{dt}\kappa_{ijkl}(t) = 4\left\{A_i\kappa_{jkl}\right\}_s + 4\left\{B_{im}\kappa_{mjkl}\right\}_s + 4\left\{C_{imn}\kappa_{mnjkl}\right\}_s + \\
&4\left\{D_{imnp}\kappa_{mnpjkl}\right\}_s + 6\sum_{\alpha=1}^{m}\left\{d_{i\alpha}d_{j\alpha}\kappa_{kl}\right\}_s \\
&\vdots \\
&\frac{d}{dt}\kappa_{i_1\cdots i_s}(t) = s\left\{E\left[Z_{i_2}^0\cdots Z_{i_s}^0 c_{i_1}^0(\mathbf{Z}(t),t)\right]\right\}_s + \\
&\frac{s(s-1)}{2}\sum_{\alpha=1}^{m}\left\{E[d_{i_1\alpha}d_{i_2\alpha}Z_{i_3}^0\cdots Z_{i_s}^0]\right\}_s
\end{aligned}
\right\} \tag{3.45}
$$

Equations for moments involve unknown expectations of non-linear functions of the state variables. Forms of these functions depend on the forms of non-linear functions $c_i(\mathbf{Z}(t))$ in the governing stochastic equations of (2.71). Equations for moments only form a closed set if $c_i(\mathbf{Z}(t))$ is a linear form in the state variables $\mathbf{Z}(t)$. When this is a polynomial of degree $r > 1$, then the equations for moments form an infinite hierarchy, i.e. the equations for the moments of up to nth order involve the moments up to and including the $n + r - 1$ order. If $c_i(\mathbf{Z}(t))$ is of other form than a polynomial in the state variables, then the pertinent expectations appearing in the equations for moments cannot be expressed explicitly in terms of moments. They must be evaluated by performing the integrals with respect to the probability density function. The exact density function is, however, unknown and the approximate, tentative, density function must be assumed. A non-Gaussian tentative density function is often assumed in form of an asymptotic, Gram-Charlier expansion, which will be discussed in subsection 3.2.2. The accuracy of the results obtained, i.e. of approximate response moments depends on how close to the exact one the assumed tentative density function is. Therefore some modifications of the tentative density function have been proposed in various problems by taking into account the physical nature of the problem, see subsection 3.3.2.

3.2.2 Asymptotic expansions of the multivariate probability density functions

Cumulants and quasi-moments

In this section we will consider the joint n-variate probability distributions. The derivations presented are equally valid for the multivariate probability distributions of a random process $Z(t)$, i.e. the joint distributions of its values $Z(t_1), Z(t_2), \ldots, Z(t_n)$ at n time points and the zero-time lag joint distributions of n-dimensional vector random process $\mathbf{Z}(t) = [Z_1(t), Z_2(t), \ldots, Z_n(t)]^T]$. The joint n-variate characteristic function of the process $\mathbf{Z}(t)$ is defined as

$$\Phi_{\{\mathbf{Z}\}}(\boldsymbol{\theta}, t) = \Phi_{\{\mathbf{Z}\}}(\theta_1, \theta_2, \ldots, \theta_n, t) = E\left[\exp\left(i \sum_{k=1}^{n} \theta_k Z_k(t)\right)\right] \tag{3.46}$$

Hence it is expressed in terms of a joint pdf as

$$\Phi_{\{\mathbf{Z}\}}(\boldsymbol{\theta}, t) = \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{n-\text{fold}} \exp\left(i \sum_{k=1}^{n} \theta_k z_k\right) f_{\{\mathbf{Z}\}}(z_1, z_2, \ldots, z_n, t) dz_1 dz_2 \cdots dz_n \tag{3.47}$$

The characteristic function can be represented as a multivariate Taylor expansion

$$\Phi_{\{\mathbf{Z}\}}(\boldsymbol{\theta}, t) = 1 + \sum_{s=1}^{\infty} \frac{i^s}{s!} \mu_{i_1 i_2 \cdots i_s}(t) \theta_{i_1} \theta_{i_2} \cdots \theta_{i_s} \tag{3.48}$$

where the summation convention over repated (dummy) indices is applied, and each of the indices assumes the values $i_k = 1, 2, \ldots n$. This is an expansion in terms of ordinary moments

$$\mu_{i_1 i_2 \cdots i_s} = E\left[Z_{i_1} Z_{i_2} \cdots Z_{i_s}\right] = \frac{1}{i^s} \frac{\partial^s \Phi_{\{\mathbf{Z}\}}(\boldsymbol{\theta}, t)}{\partial \theta_{i_1} \cdots \partial \theta_{i_s}}\bigg|_{\theta_{i_1} = \cdots = \theta_{i_s} = 0} \tag{3.49}$$

A logarithm of a characteristic function, the so-called cumulant-generating function $\ln \Phi_{\{\mathbf{Z}\}}(\boldsymbol{\theta}, t)$ has the following multivariate Taylor expansion

$$\ln \Phi_{\{\mathbf{Z}\}}(\boldsymbol{\theta}, t) = \sum_{s=1}^{\infty} \frac{i^s}{s!} \lambda_{i_1 i_2 \cdots i_s}(t) \theta_{i_1} \theta_{i_2} \cdots \theta_{i_s} \tag{3.50}$$

where $\lambda_{i_1 i_2 \cdots i_s}(t)$ are the cumulants which can be evaluated as

$$\lambda_{i_1 i_2 \cdots i_s} = \frac{1}{i^s} \frac{\partial^s \ln \Phi_{\{\mathbf{Z}\}}(\boldsymbol{\theta}, t)}{\partial \theta_{i_1} \cdots \partial \theta_{i_s}}\bigg|_{\theta_{i_1} = \cdots = \theta_{i_s} = 0} \tag{3.51}$$

Comparing (3.48) with (3.50) we have

$$\Phi_{\{\mathbf{Z}\}}(\boldsymbol{\theta}, t) = 1 + \sum_{s=1}^{\infty} \frac{i^s}{s!} \mu_{i_1 i_2 \cdots i_s}(t) \theta_{i_1} \theta_{i_2} \cdots \theta_{i_s} = \exp\left\{ \sum_{s=1}^{\infty} \frac{i^s}{s!} \lambda_{i_1 i_2 \cdots i_s}(t) \theta_{i_1} \theta_{i_2} \cdots \theta_{i_s} \right\} \tag{3.52}$$

By expanding the exponential as a Taylor series and comparing the terms of the same order in θ_{i_k}'s we can express the moments $\mu_{i_1 i_2 \cdots i_s}$ in terms of cumulants $\lambda_{i_1 i_2 \cdots i_s}$.

Inverse relations, i.e. cumulants in terms of the moments can be obtained if we take the logarithm of the characteristic function, compare it with (3.50)

$$\ln \Phi_{\{\mathbf{Z}\}}(\boldsymbol{\theta}, t) = \ln\left\{ 1 + \sum_{s=1}^{\infty} \frac{i^s}{s!} \mu_{i_1 i_2 \cdots i_s}(t) \theta_{i_1} \theta_{i_2} \cdots \theta_{i_s} \right\} = \sum_{s=1}^{\infty} \frac{i^s}{s!} \lambda_{i_1 i_2 \cdots i_s}(t) \theta_{i_1} \theta_{i_2} \cdots \theta_{i_s} \tag{3.53}$$

and next expand the logarithm as a Taylor series and compare the terms of the same order in θ_{i_k}'s in eq. (3.53).

If the first-order moments in the resulting relations are set equal to zero the expressions for the cumulants $\lambda_{i_1 i_2 \cdots i_s}(t)$ in terms of the central moments $\kappa_{i_1 i_2 \cdots i_s}(t)$ are obtained, which take the form of

$$\left.\begin{aligned}
&\lambda_i(t) = \mu_i(t) \\
&\lambda_{ij}(t) = \kappa_{ij}(t) \\
&\lambda_{ijk}(t) = \kappa_{ijk}(t) \\
&\lambda_{ijkl}(t) = \kappa_{ijkl}(t) - 3\left\{\kappa_{ij}(t)\kappa_{kl}(t)\right\}_s \\
&\lambda_{ijklm}(t) = \kappa_{ijklm}(t) - 10\left\{\kappa_{ij}(t)\kappa_{klm}(t)\right\}_s \\
&\lambda_{ijklmn}(t) = \kappa_{ijklmn}(t) - 15\left\{\kappa_{ij}(t)\kappa_{klmn}(t)\right\}_s - 10\left\{\kappa_{ijk}(t)\kappa_{lmn}(t)\right\}_s + \\
&30\left\{\kappa_{ij}(t)\kappa_{kl}(t)\kappa_{mn}(t)\right\}_s \\
&\vdots
\end{aligned}\right\} \tag{3.54}$$

The so-called quasi-moments $\beta_{i_1 i_2 \cdots i_s}(t)$ are introduced by the relation

$$\exp\left\{\sum_{s=3}^{\infty}\frac{i^s}{s!}\lambda_{i_1 i_2 \cdots i_s}(t)\theta_{i_1}\theta_{i_2}\cdots\theta_{i_s}\right\} = 1 + \sum_{s=3}^{\infty}\frac{i^s}{s!}\beta_{i_1 i_2 \cdots i_s}(t)\theta_{i_1}\theta_{i_2}\cdots\theta_{i_s} \tag{3.55}$$

The name quasi-moments is due to a similarity of the above relation with the representation (3.52).

Expanding in (3.55) the exponential as a Taylor series we can express the quasi-moments in terms of cumulants, thus

$$\left.\begin{aligned}
&\beta_{ijk}(t) = \lambda_{ijk}(t)\\
&\beta_{ijkl}(t) = \lambda_{ijkl}(t)\\
&\beta_{ijklm}(t) = \lambda_{ijklm}(t)\\
&\beta_{ijklmn}(t) = \lambda_{ijklmn}(t) + 10\left\{\lambda_{ijk}(t)\lambda_{lmn}(t)\right\}_s\\
&\beta_{ijklmnp}(t) = \lambda_{ijklmnp}(t) + 35\left\{\lambda_{ijk}(t)\lambda_{lmnp}(t)\right\}_s\\
&\beta_{ijklmnpq}(t) = \lambda_{ijklmnpq}(t) + 35\left\{\lambda_{ijkl}(t)\lambda_{mnpq}(t)\right\}_s\\
&\quad + 56\left\{\lambda_{ijk}(t)\lambda_{lmnpq}(t)\right\}_s\\
&\vdots
\end{aligned}\right\} \tag{3.56}$$

The inverse relationships are obtained by taking the logarithm of both sides of (3.55) and expanding the logarithm of the right-hand side as a Taylor series.

Asymptotic expansion of the joint pdf

The multivariate probability density function can be evaluated, in principle, as an inverse Fourier transform of a characteristic function, i.e.

$$f_{\{\mathbf{Z}\}}(\mathbf{z},t) = \frac{1}{(2\pi)^n}\underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{n\text{-fold}}\Phi_{\{\mathbf{Z}\}}(\boldsymbol{\theta},t)d\theta_1 d\theta_2\cdots d\theta_n \tag{3.57}$$

We use for the characteristic function its expression in terms of cumulants (3.52) separate the sum in the exponential and use (3.55), which yields

$$f_{\{\mathbf{Z}\}}(\mathbf{z},t) = \frac{1}{(2\pi)^n}\underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{n\text{-fold}}\exp\left(-i\sum_{k=1}^{n}\theta_k z_k\right)\exp\left\{i\lambda_{i_1}(t)\theta_{i_1} + \frac{i^2}{2}\lambda_{i_1 i_2}(t)\theta_{i_1}\theta_{i_2}\right\}\times$$
$$\left\{1 + \sum_{s=3}^{\infty}\frac{i^s}{s!}\beta_{i_1 i_2 \cdots i_s}(t)\theta_{i_1}\theta_{i_2}\cdots\theta_{i_s}\right\}d\theta_1 d\theta_2\cdots d\theta_n \tag{3.58}$$

We note that

$$\frac{1}{(2\pi)^2}\underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{n\text{-fold}}\exp\left(-i\sum_{k=1}^{n}\theta_k z_k\right)\exp\left\{i\lambda_{i_1}(t)\theta_{i_1}+\frac{i^2}{2}\lambda_{i_1 i_2}(t)\theta_{i_1}\theta_{i_2}\right\}d\theta_1 d\theta_2\cdots d\theta_n = \quad (3.59)$$

which is just the multivariate Gaussian density function with the mean values $\mu_i = \lambda_i$ and the covariances $\kappa_{ij} = \lambda_{ij}$. Next, we interchange the order of integration and summation and notice that

$$i^s\theta_{i_1}\theta_{i_2}\cdots\theta_{i_s}\exp\left(-i\sum_{k=1}^{n}\theta_k z_k\right) = (-1)^s\left(\frac{\partial}{\partial z_{i_1}}\right)\cdots\left(\frac{\partial}{\partial z_{i_1}}\right)\exp\left(-i\sum_{k=1}^{n}\theta_k z_k\right) \quad (3.60)$$

Next the order of differentiation and integration is interchanged, and the integration results in

$$(-1)^s\left(\frac{\partial}{\partial z_{i_1}}\right)\cdots\left(\frac{\partial}{\partial z_{i_s}}\right)\phi(\mathbf{z}) = H_{i_1 i_2\ldots i_s}(\mathbf{z})\varphi_n(\mathbf{z},\boldsymbol{\mu},\boldsymbol{\kappa}) \quad (3.61)$$

where $H_{i_1 i_2\ldots i_s}(\mathbf{z})$ are the multivariate, or generalized, Hermite polynomials.

Finally the multivariate probability density function is represented by the following expansion in terms of generalized Hermite polynomials, called a Gram-Charlier expansion

$$f_{\{\mathbf{Z}\}}(\mathbf{z},t) = \varphi_n(\mathbf{z},\boldsymbol{\mu},\boldsymbol{\kappa})\left\{1+\sum_{s=3}^{\infty}\frac{1}{s!}\beta_{i_1 i_2\cdots i_s}(t)H_{i_1 i_2\ldots i_s}(\mathbf{z})\right\} \quad (3.62)$$

The univariate expansion of the above form is called an Edgeworth series.

It is worthwhile to note that there can be formulated an expansion of the joint multivariate probability density function in terms of functions, or multivariate analogues of Hermite polynomials, constructed as the products of univariate Hermite polynomials, see e.g. [3.7], [3.8]. Example of application of such an expansion is given in section 3.3.

3.2.3 Closure techniques

If the centralized drift vector is given by rth order polynomials, the equations for moments up to nth order involve the moments of orders up to $n-1+r$. If we decide to truncate the hierarchy of moment equations at nth order moments, any closure consists in expressing the redundant moments of orders $n+1,\ n+2,\ldots,n-1+r$ in terms of moments of orders up to nth. Such expressions can be imposed, based on different assumptions. Let us discuss two of the most often used closure techniques.

Cumulant neglect closure technique – CNC

This technique consists in neglecting, or setting equal to zero, all the cumulants of orders higher than n. Closure approximations, i.e. the expressions for moments of orders $n+1,\ n+2,\ldots,n-1+r$ in terms of lower order moments are obtained by means of the relationships between the moments and cumulants, in which the cumulants of orders higher than n are set equal to zero cf. (3.54).

It should be noted, however, that , as Marcinkiewicz theorem [3.9] states it, any stochastic process can have either first- and second-order cumulants non-zero (Gaussian process) or all cumulants non-zero. Hence, strictly speaking, no stochastic process exists whose cumulants up to the order $n > 2$ are non-zero and all others are zero. In other words the moments evaluated with the help of cumulant-neglect closure approximations are not exact, or actual (true), moments of any stochastic process. They may only be regarded as approximations to the actual moments.

The lowest-order cumulant neglect closure, obtained by neglecting the cumulants of orders higher than two, is the so-called Gaussian closure. Then the hierarchy of moments equations is truncated at 2nd order moments.

The problem can be best explained at an example. Consider the system with cubic non-linearity. Let us truncate the hierarchy of moment equations at 4th order moments. Hence the redundant moments are of 5th and 6th order. The closure approximation for 5th and 6th order central moments, resulting from neglecting the 5th and 6th order cumulants are, see (3.54)

$$\left.\begin{aligned}
&\kappa_{ijklm}(t) = 10\left\{\kappa_{ij}(t)\kappa_{klm}(t)\right\}_s \\
&\kappa_{ijklmn}(t) = 15\left\{\kappa_{ij}(t)\kappa_{klmn}(t)\right\}_s + 10\left\{\kappa_{ijk}(t)\kappa_{lmn}(t)\right\}_s - \\
&30\left\{\kappa_{ij}(t)\kappa_{kl}(t)\kappa_{mn}(t)\right\}_s
\end{aligned}\right\} \tag{3.63}$$

For the formulation of the Gaussian closure for the system with 3rd and 5th order polynomial non-linearity see Example Problem 3.2.

Quasi-moment neglect closure technique – QMNC

This technique consists in neglecting the quasi-moments of orders higher than n. As seen from (3.61) this is equivalent to truncating the series (3.61) at the order $s = 3$. Actual the QMNC will give the same solution for the unprovided moments as if these were evaluated by the truncated series (3.62). Since the quasi-moments up to the 5th order are equal to the cumulants, the QMNC gives the same results as a CNC up to the 5th order. The differences begin with the 6th order. Setting the 6th order quasi-moment β_{ijklmn} equal to zero in (3.56) yields

$$\lambda_{ijklmn} = -10\left\{\lambda_{ijk}(t)\lambda_{lmn}(t)\right\}_s \tag{3.64}$$

Using in (3.64) the expression for the 6th order cumulant (3.54) results in the closure approximation for the 6th order central moments

$$\kappa_{ijklmn}(t) = 15\left\{\kappa_{ij}(t)\kappa_{klmn}(t)\right\}_s - 30\left\{\kappa_{ij}(t)\kappa_{kl}(t)\kappa_{mn}(t)\right\}_s \tag{3.65}$$

which differs from (3.63) by the term involving 3rd order moments.

Corresponding forms of the Gram-Charlier expansion

Let us now look at the forms of Gram-Charlier expansions corresponding with nth order cumulant neglect and quasi-moment neglect closures.

The Gram-Charlier expansion corresponding with the nth order quasi-moment neglect closure involves only the terms of orders up to nth. Such a truncated expansion contains obviously a finite number of terms. The closure approximations for moments will be obtained exactly the same if the central moments are evaluated by performing the integration with respect to the tentative joint pdf assumed in the from of such a truncated expansion.

The Gram-Charlier expansion corresponding with the nth order CNCT should be consequently obtained by setting in it the cumulants above the nth order equal to zero. As it is seen from the relationships between the quasi-moments and the cumulants, the expansion coefficients above the nth order do not vanish, they are just expressed as different products of lower-order cumulants, cf. (3.55). Consequently all the higher order generalized Hermite polynomials are present in the expansion, and the expansion is still infinite. Evaluating the central moments by performing the integration with respect to such a tentative joint pdf yields the same results as the CNCT. It should be noted, however, that while evaluating the redundant moments up to the order $(n-1+r)$ it suffices to keep in the expansion the terms with the Hermite polynomials up to the order $n-1+r$ only, because the integration with respect to the Hermite polynomials of orders higher than $n-1+r$ then gives zero.

3.3 Examples of SDOF non-linear non-hysteretic and hysteretic systems

Example 3.1: Duffing oscillator

Consider the Duffing oscillator (1.87) subjected to a Gaussian white noise excitation $\{F(t), t \in [0, \infty[\}$ with the auto-spectral density function S_0. The stationary variance of the displacement and the velocity of the corresponding linear oscillator is known to be, Nielsen [3.10]

$$\sigma^2_{Y,0} = \frac{\pi S_0}{2\zeta\omega_0^3} \quad , \quad \sigma^2_{\dot{Y},0} = \frac{\pi S_0}{2\zeta\omega_0} \tag{3.66}$$

For ease the displacement and the time are made non-dimensional as follows

$$\tau = \omega_0 t \tag{3.67}$$

$$X(t) = \frac{Y(t)}{\sigma_{Y,0}} \tag{3.68}$$

Equation (1.87) can then be written

$$\ddot{X} + 2\zeta\dot{X} + X + \kappa X^3 = 2\sqrt{\zeta}\dot{W}(\tau) \tag{3.69}$$

$$\kappa = \sigma_{Y,0}^2 \varepsilon \tag{3.70}$$

where dot means differentiation with respect to the non-dimensional time τ. $\{\dot{W}(\tau), \tau \in [0, \infty[\}$ is a unit intensity white noise, i.e. its auto-covariance function and auto-spectral density function become

$$\kappa_{\dot{W}\dot{W}}(u) = E[\dot{W}(\tau)\dot{W}(\tau + u)] = \delta(u) \tag{3.71}$$

$$S_{\dot{W}\dot{W}}(\omega) = \frac{1}{2\pi} \tag{3.72}$$

From (3.66) and (3.72) follows that the indicated normalization are tantamount to normalize the stationary variances of the linear oscillator to $\sigma_{Y,0}^2 = \sigma_{\dot{Y},0}^2 = 1$.

The stationary solution for the joint probability density function of $Z^T(\tau) = [X(\tau), \dot{X}(\tau)] = [Z_1, Z_2]$ follows from (3.30)

$$\begin{aligned} f_{\{\mathbf{Z}\}}(z_1, z_2) &= \frac{C}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z_2^2 - \left(\frac{z_1^2}{2} + \kappa\frac{z_1^4}{4}\right)\right) \\ &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z_2^2\right) \cdot C \exp\left(-\left(\frac{z_1^2}{2} + \kappa\frac{z_1^4}{4}\right)\right) \end{aligned} \tag{3.73}$$

As seen $Z_1(t)$ and $Z_2(t)$ are stochastically independent and $Z_2(t) \sim N(0,1)$. Obviously, the stationary distribution only exist for $\kappa > 0$, i.e. for hardening springs. The normalization constant becomes, Wu and Lin [3.11]

$$C = \frac{1}{\sqrt{2\kappa}} \exp\left(\frac{1}{8\kappa}\right) K_{\frac{1}{4}}\left(\frac{1}{8\kappa}\right) \tag{3.74}$$

where $K_{\frac{1}{4}}$ is the modified Bessel function of the order $\frac{1}{4}$. The variance $E[Z_1^2]$ can next be evaluated analytically, Wu and Lin [3.11]

$$E[Z_1^2] = C\int_{-\infty}^{\infty} z_1^2 \exp\left(-\frac{z_1^2}{2} - \frac{1}{4}\frac{z_1^4}{4}\right) dz_1 = C\frac{\sqrt{\pi}}{2}\left(\frac{\kappa}{2}\right)^{-\frac{3}{4}} \exp\left(\frac{1}{8\kappa}\right) D_{-\frac{3}{2}}\left(\frac{1}{\sqrt{2\kappa}}\right) \tag{3.75}$$

where $D_{-\frac{3}{2}}$ is the parabolic cylinder function of the order $-\frac{3}{2}$.

In the following the exact result (3.75) will be used to check the accuracy of the cumulant neglect closure scheme for the stationary response variance of the oscillator. In order to do this (3.69) is first replaced by the following coupled equivalent Itô stochastic differential equations

$$\left.\begin{aligned} dZ_1 &= Z_2 d\tau \\ dZ_2 &= (-2\zeta Z_2 - Z_1 - \kappa Z_1^3)d\tau + 2\sqrt{\zeta}dW(\tau) \end{aligned}\right\} \tag{3.76}$$

The moment equations (3.45) then become

$$\left.\begin{aligned} \frac{d}{d\tau}E[Z_1] &= E[Z_2] \\ \frac{d}{d\tau}E[Z_2] &= -2\zeta E[Z_2] - E[Z_1] - \kappa E[Z_1^3] \end{aligned}\right\} \tag{3.77}$$

$$\left.\begin{aligned}
&\frac{d}{d\tau}E[Z_1^2] = 2E[Z_1Z_2] \\
&\frac{d}{d\tau}E[Z_1Z_2] = E[Z_2^2] - 2\zeta E[Z_1Z_2] - E[Z_1^2] - \kappa E[Z_1^4] \\
&\frac{d}{d\tau}E[Z_2^2] = -4\zeta E[Z_2^2] - 2E[Z_1Z_2] - 2\kappa E[Z_1^3Z_2] + 4\zeta
\end{aligned}\right\} \tag{3.78}$$

$$\left.\begin{aligned}
&\frac{d}{d\tau}E[Z_1^3] = 3E[Z_1^2Z_2] \\
&\frac{d}{d\tau}E[Z_1^2Z_2] = 2E[Z_1Z_2^2] - 2\zeta E[Z_1^2Z_2] - E[Z_1^3] - \kappa E[Z_1^5] \\
&\frac{d}{d\tau}E[Z_1Z_2^2] = E[Z_2^3] - 4\zeta E[Z_1Z_2^2] - 2E[Z_1^2Z_2] - 2\kappa E[Z_1^4Z_2] + 4\zeta E[Z_1] \\
&\frac{d}{d\tau}E[Z_2^3] = -6\zeta E[Z_2^3] - 3E[Z_1Z_2^2] - 3\kappa E[Z_1^3Z_2^2] + 12\zeta E[Z_2]
\end{aligned}\right\} \tag{3.79}$$

$$\left.\begin{aligned}
&\frac{d}{d\tau}E[Z_1^4] = 3E[Z_1^3Z_2] \\
&\frac{d}{d\tau}E[Z_1^3Z_2] = 3E[Z_1^2Z_2^2] - 2\zeta E[Z_1^3Z_2] - E[Z_1^4] - \kappa E[Z_1^6] \\
&\frac{d}{d\tau}E[Z_1^2Z_2^2] = 2E[Z_1Z_2^3] - 4\zeta E[Z_1^2Z_2^2] - 2E[Z_1^3Z_2] - 2\kappa E[Z_1^5Z_2] + 4\zeta E[Z_1^2] \\
&\frac{d}{d\tau}E[Z_1Z_2^3] = E[Z_2^4] - 6\zeta E[Z_1Z_2^3] - 3E[Z_1^2Z_2^2] - 3\kappa E[Z_1^4Z_2^2] + 12\zeta E[Z_1Z_2] \\
&\frac{d}{d\tau}E[Z_2^4] = -8\zeta E[Z_2^4] - 4E[Z_1Z_2^3] - 4\kappa E[Z_1^3Z_2^3] + 24\zeta E[Z_2^2]
\end{aligned}\right\} \tag{3.80}$$

In the stationary state, the derivatives of these moments must vanish and the indicated differential equations reduce to algebraic equations. The restoring force $Z_1 + \kappa Z_1^3$ is an odd function of Z_1. In the stationary state, where the response from any initial values has been dissipated, there will then be no preference for positive or negative displacements. Hence, it can be argued that $E[Z_1] = 0 \Rightarrow E[Z_2] = \frac{d}{d\tau}E[Z_1] = 0$. The moment equations (3.78) - (3.80) are then equivalent to the corresponding centralized moment equations (3.45). First consider cumulant neglect closure at the order $n = 2$. Then all cumulants above $n = 2$ are set to zero resulting in, cf. (3.54)

$$\left.\begin{aligned}
&E[Z_1^3] = 0 \\
&E[Z_1^4] = 3(E[Z_1^2])^2 \\
&E[Z_1^3Z_2] = 3E[Z_1^2]E[Z_1Z_2]
\end{aligned}\right\} \tag{3.81}$$

Insertion of (3.81) into (3.78) provides

$$\left.\begin{aligned}
&0 = 2E[Z_1Z_2] \\
&0 = E[Z_2^2] - 2\zeta E[Z_1Z_2] - E[Z_1^2] - \kappa \cdot 3(E[Z_1^2])^2 \\
&0 = -4\zeta E[Z_2^2] - 2E[Z_1Z_2] - 2\kappa \cdot 3E[Z_1^2]E[Z_1Z_2] + 4\zeta
\end{aligned}\right\} \Rightarrow \tag{3.82}$$

$$\left.\begin{aligned} 0 &= E[Z_1 Z_2] \\ 0 &= E[Z_2^2] - E[Z_1^2] - 3\kappa(E[Z_1^2])^2 \\ 0 &= -4\zeta(1 - E[Z_2^2]) \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} E[Z_1^2] &= \frac{-1+\sqrt{1+12\kappa}}{6\kappa} \\ E[Z_2^2] &= 1 \\ E[Z_1 Z_2] &= 0 \end{aligned}\right\} \tag{3.83}$$

Cumulant neglect closure at the order $n = 2$ then provide the exact solutions $E[Z_1^2] = 1$, $E[Z_1 Z_2] = 0$ and the indicated approximate solutions for the variance $A = E[Z_1^2]$, which is obtained as the positive solutions of the quadratic equation $3\kappa A^2 - A + 1 = 0$. In the same way it can be shown, that the cumulant neglect closure solutions for $A = E[Z_1^2]$ at the 4th order and the 6th order are obtained as the positive solutions of the following cubic and quartic equations, Wu and Lin [3.11]

$$30\kappa^2 A^3 + 15\kappa A^2 + (1 - 12\kappa)A - 1 = 0 \tag{3.84}$$

$$714\kappa^3 A^4 + 420\kappa^2 A^3 + (63 - 336\kappa)\kappa A^2 + (1 - 90\kappa)A - (1 - 30\kappa) = 0 \tag{3.85}$$

The solution (3.83) for $E[Z_1^2]$ is identical with the so-called Gaussian closure result, obtained by calculating the expectations on the right-hand side of (3.82) assuming Z_1 and Z_2 to be jointly normal distributed with the covariance $E[Z_1^2]$, $E[Z_2^2]$ and $E[Z_1 Z_2]$. The indicated solutions (3.83), (3.84) and (3.85) have been checked with the exact results as a function of κ as shown in figure 3.1. The most important observation is that cumulant neglect closure technique underestimates the stationary variance and that the improvement from the 4th to the 6th order closure is modest. Hence, it can be concluded that if the cumulant neglect closure scheme converge to the exact result at all, the rate of convergence is at least very slow.

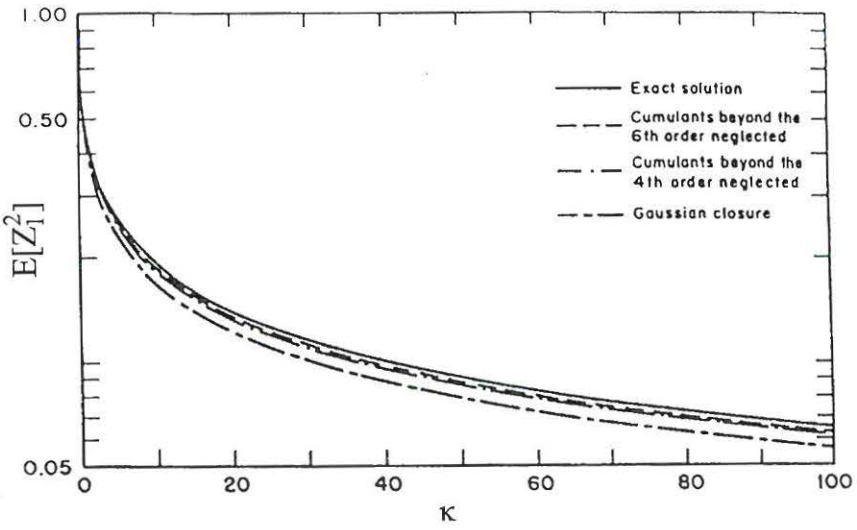

Fig. 3.1. Stationary variance of a Duffing oscillator under external white noise excitation as a function of the non-linearity parameter κ, Wu and Lin [3.11].

Example 3.2: SDOF hysteretic system

Consider the bilinear oscillator (1.92), (1.95) subject to white noise excitation. In the considered non-linear formulation as defined by (1.90), (1.94) this system can be represented as the following equivalent

system of 1st order Itô stochastic differential equations.

$$\left.\begin{aligned} dZ_1 &= Z_2 d\tau \\ dZ_2 &= (-2\zeta Z_2 - \alpha Z_1 - (1-\alpha)Z_3)d\tau + 2\sqrt{\zeta}dW(\tau) \\ dZ_3 &= \kappa(Z_2, Z_3)Z_2 d\tau \end{aligned}\right\} \tag{3.86}$$

where

$$k(Z_2, Z_3) = (1 - (1 - 1(-Z_2))\mathbf{1}(-Z_3 - z_0) - (1 - \mathbf{1}(Z_2))\mathbf{1}(-Z_3 - z_0))) \tag{3.87}$$

where $Z_3(\tau) = \frac{Q(\tau)}{\sigma_Y}$ is a non-dimensional hysteretic component and $z_0 = \frac{q_0}{\sigma_Y}$ is a non-dimensional yield-level. In the non-dimensional formulation z_0 becomes a measure of the strength, or heaviness, of the excitation. Small values of z_0 correspond to heavy excitation (since $\sigma_Y >> z_0$) and the large values, to light excitation. A value $z_0 = 1$ ($\sigma_Y = q_0$) may be classified as medium to large excitation. $\kappa(Z_2, Z_3)$ can be interpreted as a non-linear, state-dependent spring stiffness on the hysteretic component Z_3. $\kappa = 0$ on the plastic branches, and $\kappa = 1$ on the elastic branches as shown in figure 3.2.

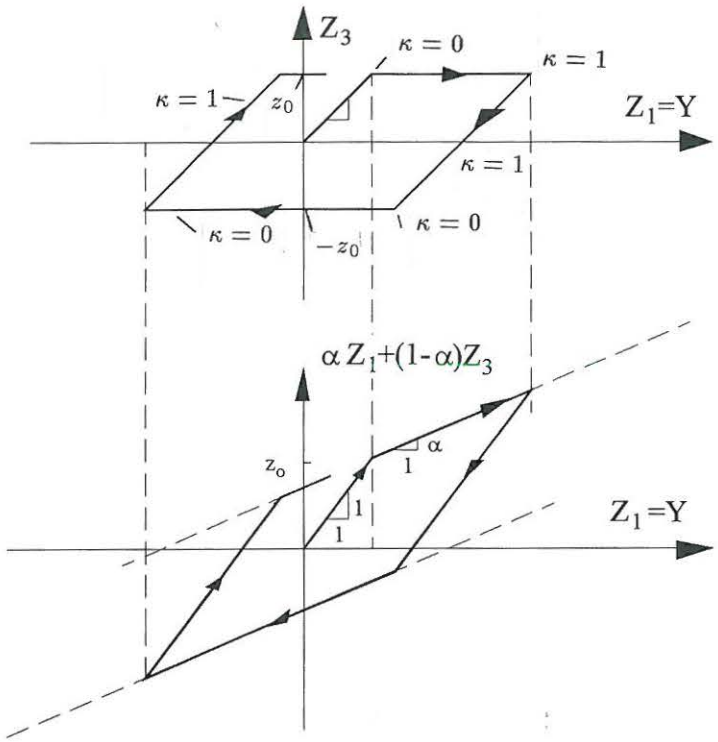

Fig. 3.2. The bilinear non-dimensional hysteretic restoring component Z_3 and the total restoring force $\alpha Z_1 + (1-\alpha)Z_3$ as function of the system deflection Z_1, Nielsen et al. [3.12].

Obviously, there is a finite probability of attaining the plastic states $Z_3 = \pm z_0$. This means that the marginal probability density function $f_{Z_3}(z_3, t)$ for the hysteretic component must be of the mixed type with a continuous distribution for $Z_3 \in]-z_0, z_0[$ and discrete probabilities at $Z_3 = \pm z_0$. From the physical reasoning $f_{Z_3}(z_3, t)$ is then assumed on the form

$$f_{Z_3}(z_3, t) = f^0_{Z_3}(z_3) + \delta(z_3 - z_0)\int_{z_0}^{\infty} f^0_{Z_3}(u)du + \delta(z_3 + z_0)\int_{-\infty}^{-z_0} f^0_{Z_3}(u)du \tag{3.88}$$

where $f^0_{Z_3}(u)$ is continuous. As seen, the discrete probability mass concentrations at $Z_3 = \pm z_0$ are $\int_{z_0}^{\infty} f^0_{Z_3}(u)du$ and $\int_{-\infty}^{-z_0} f^0_{Z_3}(u)du$, respectively. The form of pdf (3.88) was first suggested by Minai and Suzuki [3.13] and will be referred to as a Minai an Suzuki modification of the continous type pdf $f^0_{Z_3}(u)$ in what follows.

In order to close the system of moment equations in this case a tentative joint probability density for the state vector with the indicated Minai-Suzuki modification is suggested.

$$f_{Z_1Z_2Z_3}(z_1,z_2,z_3) = f^0_{Z_1Z_2Z_3}(z_1,z_2,z_3)+$$
$$\delta(z_3-z_0)\int_{z_0}^{\infty} f^0_{Z_1Z_2Z_3}(z_1,z_2,u)du + \delta(z_3+z_0)\int_{-\infty}^{-z_0} f^0_{Z_1Z_2Z_3}(z_1,z_2,u)du \tag{3.89}$$

For $f^0_{Z_1Z_2Z_3}(z_1,z_2,z_3)$ the following Gram-Charlier expansion is applied, cf. Dashevski [3.14], Assaf and Zirkle [3.15].

$$f^0_{Z_1Z_2Z_3}(z_1,z_2,z_3) = \frac{1}{\sigma^0_{Z_1}\sigma^0_{Z_2}\sigma^0_{Z_3}}\varphi\left(\frac{z_1}{\sigma^0_{Z_1}}\right)\varphi\left(\frac{z_2}{\sigma^0_{Z_2}}\right)\varphi\left(\frac{z_3}{\sigma^0_{Z_3}}\right)\cdot$$
$$\sum_{i+j+k=0}^{N} c^0_{ijk}H_i\left(\frac{z_1}{\sigma^0_{Z_1}}\right)H_j\left(\frac{z_2}{\sigma^0_{Z_2}}\right)H_k\left(\frac{z_3}{\sigma^0_{Z_3}}\right) \tag{3.90}$$

where

$$H_i(x) = \sum_{\alpha=0}^{[i/2]}(-1)^\alpha a_{i,\alpha}x^{i-2\alpha}, \quad a_{i,\alpha} = \frac{i!2^{-\alpha}}{\alpha!(i-2\alpha)!} \tag{3.91}$$

$$c^0_{ijk} = \frac{1}{i!j!k!}E\left[H_i\left(\frac{Z_1}{\sigma^0_{Z_1}}\right)H_j\left(\frac{Z_2}{\sigma^0_{Z_2}}\right)H_k\left(\frac{Z_3}{\sigma^0_{Z_3}}\right)\right]_0 \tag{3.92}$$

$\varphi(x)$ is the probability density function of a standardized normal variate, $H_i(\cdot)$ is the Hermite polynomial of the ith degree and $[\cdot]$ denotes the integer part. The expansion (3.90) implicitly assumes the response of the oscillator to be zero mean, i.e. $E[Z_1] = E[Z_2] = E[Z_3] = 0$

The expectation $E[\cdot]_0$ in (3.92) are with respect to the joint pdf $f^0_{Z_1Z_2Z_3}(z_1,z_2,z_3)$ which equally fulfills the zero-mean condition. It then follows from (3.92) that $c^0_{ijk} = 0, i+j+k$ odd. Moreover, it follows directly from (3.92) that $c^0_{000} = 1$ and $c^0_{200} = c^0_{020} = c^0_{002} = 0$. The remaining non-trivial free parameters of the tentative pdf (3.89) including $\sigma^0_{Z_1}$, $\sigma^0_{Z_2}$ and $\sigma^0_{Z_3}$ are determined from the following joint moment relations, as derived from (3.89) and (3.90).

$$E[Z_1^lZ_2^mZ_3^n] = (\sigma^0_{Z_1})^l(\sigma^0_{Z_2})^m(\sigma^0_{Z_3})^n\sum_{i+j+k=0}^{N} c^0_{ijk}r_{l,i}r_{m,jr_{n,k}}s_{n,k}(\beta) \tag{3.93}$$

where $\beta = \frac{z_0}{\sigma^0_{Z_3}}$ and

$$r_{l,i} = \int_{-\infty}^{\infty} u^l H_i(u)\varphi(u)du = \begin{cases} 0, & (i>l)\vee(l-i \text{ odd}) \\ i!a_{l,(l-i)/2}, & (i\le l)\wedge(l-i \text{ even}) \end{cases} \tag{3.94}$$

$$s_{n,k}(\beta) = \int_{-\beta}^{\beta} u^n H_k(u)\varphi(u)du + \beta^n \int_{\beta}^{\infty} H_k(u)\varphi(u)du + (-\beta)^n \int_{-\infty}^{-\beta} H_k(u)\varphi(u)du =$$

$$\begin{cases} 0 & , \quad (n-k \text{ odd}) \\ r_{n,k} + 2\sum_{\alpha=0}^{[k/2]} (-1)^{\alpha} a_{k,\alpha}(\beta^n t_{k-2\alpha}(\beta) - t_{n+k-2\alpha}(\beta)) & , \quad (n-k \text{ even}) \end{cases} \tag{3.95}$$

$$t_k(\beta) = \int_{\beta}^{\infty} u^k \varphi(u)du \tag{3.96}$$

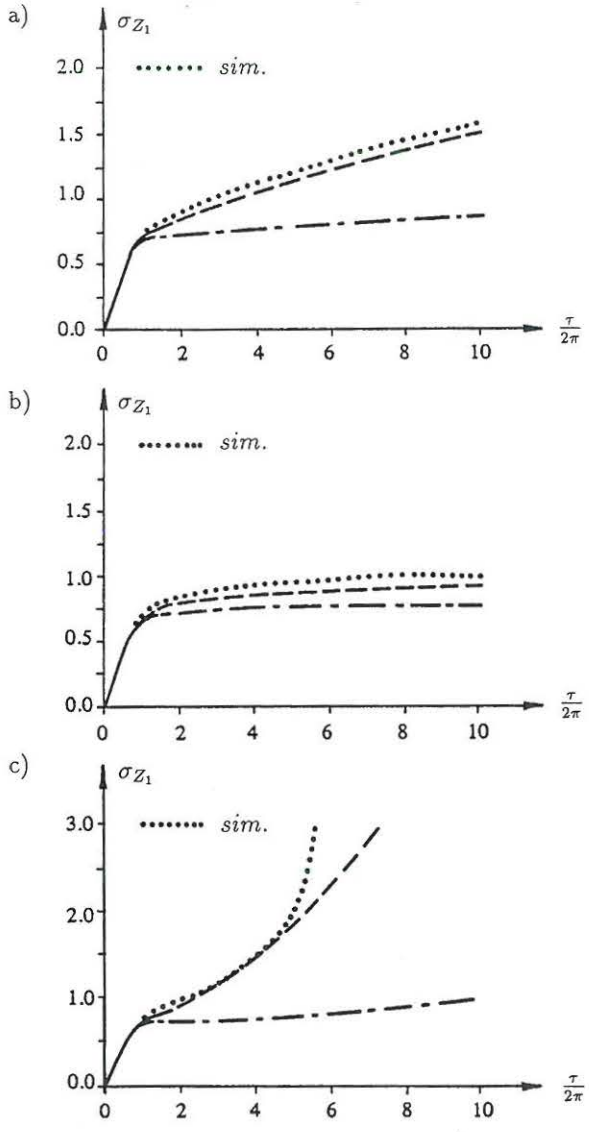

Fig. 3.3. Standard deviation $\sigma_{Z_1}(\tau)$ of non-dimensional displacement response, $\zeta = 0.05$, $z_0 = 1.0$. – – –: Minai-Suzuki modification, Gram-Charlier expansion at order $N = 4$. – · – · –: Gaussian closure. $N = 2$. a) $\alpha = 0$, b) $\alpha = 0.1$ and c) $\alpha = -0.1$.

t_k can be explicitly expressed in terms of $\varphi(\beta)$ and $\Phi(\beta)$, where $\Phi(\beta)$ is the distribution function of a standardized normal variate. $\sigma^0_{Z_1} = \sigma_{Z_1}$ and $\sigma^0_{Z_2} = \sigma_{Z_2}$ follow directly from (3.93). For the case $N = 4$ σ_{Z_3} is determined from the transcendent equation

$$E[Z_3^2] = (\sigma^0_{Z_3})^2 s_{2,0} \left(\frac{z_0}{\sigma_{Z_3^0}} \right) \tag{3.97}$$

Note that (3.97) is obtained from (3.93) by setting $c^0_{004} = 0$. This restriction has been imposed in order to prevent negative side loops of the approximate joint pdf and has the consequence that it cannot be calibrated to $E[Z_3^4]$.

The stochastic response of the system (3.86) is searched for in case the system starts from rest with the initial values $(Z_1, Z_2, Z_3) = (0, 0, 0)$. The results for the non-stationary variation of the displacement standard deviation $\sigma_{Z_1}(t)$ obtained from the present theory using a Minai-Suzuki modified Gram-Charlier expansion truncated at the order $N = 4$, have been compared in fig. 3.3 with those obtained by Monte-Carlo simulation and by a Gaussian closure. The considered system has the damping ratio $\zeta = 0.05$, the non-dimensional yield level $z_0 = 1$ and the considered values of the post to primary stiffness ratio are $\alpha = 0.0$, $\alpha = 0.1$ and $\alpha = -0.1$. Positive values of α correspond to less non-linear systems. Hence, it is quite natural, that equivalent linearization is doing best in the case $\alpha = 0.1$. For the cases where $\alpha \leq 0.0$ the oscillator will not have a stationary distribution, but the displacement variance grows to infinity, whereas the Gaussian closure predicts a stationary variance in this case as seen from figs. 3.3 a and 3.3 c.

The overall conclusion from this example is that if any success in prediction of the response variance of heavily non-linear hysteretic oscillators is to be made, a closure scheme must be used which displays the physical nature of the system behaviour, such as the indicated Minai-Suzuki modification. Further, the nonstationary variance drift for $\alpha \leq 0$ is hidden in the 4th order moments.

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3.5 Example problems

3.1 Using exact response probability density function of the Duffing oscillator, evaluate the stationary velocity and displacement response variances.

3.2 Derive the equations for response moments up to the fourth order of the general non-linear oscillator the under a Gaussian white noise excitation and:

a) Discuss the stationary solution in the non-linear case.

b) In the case of a Duffing oscillator apply the following closure approximations:

- Gaussian closure,
- cumulant-neglect closure,
- quasi-moment closure.

CHAPTER 4

RANDOM PULSE TRAINS DRIVEN BY DIFFERENT STOCHASTIC POINT PROCESSES

4.1 Stochastic point processes

4.1.1 Specification of a random counting process

Let us denote by $N(t)$ a random counting process, or a random variable, specifying the number of events (or time points) in an interval $[0, t[$. Strictly speaking, an additional assumption $\Pr\{N(0) = 0\} = 1$ should also be imposed. Here, however the occurrences of events are not assumed to be independent, since they are not assumed to be Poisson distributed. The increment $dN(t)$ of the counting process during an infinitesimal time interval is defined as in the section 1.1.2.

The point process is *regular* or *orderly*, if the probability governing the counting measure satisfies the following condition

$$\sum_{k\geq 2} \Pr\{dN(t) = k\} = \mathrm{O}(dt^2) \tag{4.1}$$

which means that in the infinitesimal time interval there can only occur, with non-zero probability, one event or no event.

Let us choose from the interval $[0, t[$ the disjoint infinitesimal time intervals $[t_i, t_i + dt_i[$, $i = 1, 2, \ldots, n$. Product density functions are defined as follows, Srinivasan [4.1]

$$f_n(t_1, t_2, \ldots, t_n)\, dt_1 \cdots dt_n = E\left[\prod_{i=1}^{n} dN(t_i)\right] \tag{4.2}$$

Equivalently, if the point process is regular, the nth degree product density function $f_n(t_1, \ldots, t_n)$ represents the probability that one event occurs in each of disjoint intervals $[t_i, t_i + dt_i[$, irrespectively of other events in the interval $[0, t[$, thus

$$f_n(t_1, t_2, \ldots, t_n) dt_1 \cdots dt_n = \Pr\{dN(t_1) = 1 \wedge \ldots \wedge dN(t_n) = 1\} =$$

$$\Pr\{\bigwedge_{i=1}^{n} dN(t_i) = 1\}\ ,\ t_1 \neq t_2 \neq \cdots \neq t_n \tag{4.3}$$

In what follows we confine our attention to regular or orderly point processes. In particular

$$\Pr\{dN(t) = 1\} = f_1(t) dt \tag{4.4}$$

where $f_1(t)$ is the product density of degree one. The regularity assumption (4.1) implies that

$$\Pr\{dN(t) = 0\} = 1 - f_1(t)dt + \mathrm{O}(dt^2) \tag{4.5}$$

$$E\left[dN(t)\right] = f_1(t)dt + \mathrm{O}(dt^2) \tag{4.6}$$

and

$$E\left[\{dN(t)\}^n\right] = f_1(t)dt + \mathrm{O}(dt^2) \tag{4.7}$$

for arbitrary n. Product density of degree one $f_1(t)$ represents the mean rate of occurrence of events (mean arrival rate). It should be noted that $f_1(t)$ is not a probability density; its integration over the whole time interval $[0, t[$ yields an expected number of events in this interval, which usually is not equal to one

$$\int_0^t f_1(\tau)d\tau = \int_0^t E\left[dN(\tau)\right] = E\left[\int_0^t dN(\tau)\right] = E[N(t)] \tag{4.8}$$

Product density of degree two, satisfying the relationship

$$f_2(t_1, t_2)dt_1 dt_2 = E\left[dN(t_1)dN(t_2)\right] \quad , \quad t_1 \neq t_2 \tag{4.9}$$

specifies the correlation between arrival rates at two different time instants t_1, t_2 (or the correlation of increments of the counting measure $N(t)$ on disjoint infinitesimal time intervals).

If k out of n time instants are set equal, i.e. $t_{j_1} = t_{j_2} = \cdots = t_{j_k}$, or k out of n infinitesimal intervals all overlap, the product density of degree n degenerates to $(n - k + 1)$th degree product density, thus

$$E\left[\prod_{i=1}^{n} dN(t_i)\right]\Bigg|_{t_{j_1}=\cdots=t_{j_k}} = E\left[\prod_{i=1}^{n-k} dN(t_i)\{dN(t_{j_1})\}^k\right] =$$

$$f_{n-k+1}(t_1, \cdots, t_{n-k}, t_{j_1})dt_1 \cdots dt_{n-k}dt_{j_1} \quad , \quad i \neq j_r \quad , \quad r = 1, \ldots, k \tag{4.10}$$

For example

$$E\left[dN(t_1)dN(t_2)\right]\Big|_{t_1=t_2} = E\left[\left\{dN(t_1)\right\}^2\right] = f_1(t)dt \tag{4.11}$$

Joint density function, defined as

$$\pi_n(t_1, t_2, \ldots, t_n)dt_1 dt_2 \cdots dt_n = \Pr\left\{\bigwedge_{i=1}^{n} dN(t_i) = 1 \wedge N(t) = n\right\} \tag{4.12}$$

specifies the probability that one event occurs in each of disjoint intervals $[t_i, t_i + dt_i[$ and there is no other events in the whole time interval $[0, t[$, i.e. that there is exactly $N(t) = n$ events.

The following relationships between the product density and the joint density functions hold, cf. Srinivasan [4.1]

$$f_k(t_1, \ldots, t_k) = \sum_{n=k}^{\infty} \frac{1}{(n-k)!} \underbrace{\int_0^t \cdots \int_0^t}_{(n-k)\text{–fold}} \pi_n(t_1, \ldots, t_k, t_{k+1}, \ldots, t_n) dt_{k+1} \cdots dt_n \quad (4.13)$$

$$\pi_k(t_1, \ldots, t_k) = \sum_{n=k}^{\infty} \frac{(-1)^{n-k}}{(n-k)!} \underbrace{\int_0^t \cdots \int_0^t}_{(n-k)\text{–fold}} f_n(t_1, \ldots, t_k, t_{k+1}, \ldots, t_n) dt_{k+1} \cdots dt_n \quad (4.14)$$

The probability that exactly n events occur in the time interval $[0, t[$ is evaluated as, Srinivasan [4.1]

$$\Pr\{N(t) = n\} = \frac{1}{n!} \underbrace{\int_0^t \cdots \int_0^t}_{n\text{–fold}} \pi_n(t_1, t_2, \ldots, t_n) dt_1 dt_2 \cdots dt_n \quad (4.15)$$

Moreover the correlation functions of the nth degree are defined in terms of product densities as, Stratonovich [4.2]

$$\left.\begin{aligned} g_1(t) &= f_1(t) \\ g_2(t_1, t_2) &= f_2(t_1, t_2) - f_1(t_1) f_1(t_2 \\ g_3(t_1, t_2, t_3) &= f_3(t_1, t_2, t_3) - 3\{f_1(t_1) f_2(t_2, t_3)\}_s + 2 f_1(t_1) f_1(t_2) f_1(t_3) \end{aligned}\right\} \quad (4.16)$$

where $\{\cdots\}_s$ denotes the symmetrizing operation, i.e. the arithmetic mean of all terms similar to the one in brackets and obtained by all possible permutations of t_1, t_2, t_3. For example

$$\{f_1(t_1) f_2(t_2, t_3)\}_s = \frac{1}{3}\Big(f_1(t_1) f_2(t_2, t_3) + f_1(t_2) f_2(t_1, t_3) + f_1(t_3) f_2(t_1, t_2)\Big) \quad (4.17)$$

4.1.2 Poisson process

Poisson process is a special case of a point process, whose increments $dN(t)$ defined on disjoint time intervals dt are independent. The non-homogeneous Poisson process is completely characterized by its first-order product density

$$f_1(t) = \nu(t) \quad (4.18)$$

which is the *intensity* of the Poisson process.

Higher-order correlation functions are equal zero

$$g_n(t_1, \ldots, t_n) = 0 \quad , \quad n > 1 \tag{4.19}$$

Due to the independence of increments (4.2) becomes

$$f_n(t_1, \ldots, t_n) = \prod_{i=1}^{n} \nu(t_i) \tag{4.20}$$

Substituting (4.20) in (4.14) and in (4.15) one obtains, respectively, cf. e.g. [4.3, 4.4]

$$\pi_n(t_1, \ldots, t_n) = \prod_{i=1}^{n} \nu(t_i) \exp\left(- \int_0^t \nu(\tau) d\tau \right) \tag{4.21}$$

$$\Pr\{N(t) = n\} = \frac{1}{n!} \left(\int_0^t \nu(\tau) d\tau \right)^n \exp\left(- \int_0^t \nu(\tau) d\tau \right) \tag{4.22}$$

For a homogeneous Poisson process $(\nu(t) = \nu = \text{const.})$ one obtains the following expressions

$$f_n(t_1, \ldots, t_n) = \nu^n \tag{4.23}$$

$$\pi_n(t_1, \ldots, t_n) = \nu^n \exp(-\nu t) \tag{4.24}$$

$$\Pr\{N(t) = n\} = \frac{(\nu t)^n}{n!} \exp(-\nu t) \tag{4.25}$$

4.1.3 Renewal processes

The renewal process can be defined as a sequence of random time points $t_1, t_2, \ldots, t_n$ on the positive real line, such that

$$\left.\begin{aligned} t_i - t_{i-1} &= T_i \quad , \quad i = 2, 3, \ldots \\ t_1 &= T_1 \end{aligned}\right\} \tag{4.26}$$

where the time intervals $\{T_i,\ i = 2, 3, \ldots\}$ between the successive points, called *interarrival times* are the positive, independent and identically distributed random variables. The point process is called an *ordinary renewal process* if the time T_1 measured from the origin to the first event has the same distribution as other time intervals T_i. This means that the time origin is placed at the instant of 0th, or initial, event which is not

counted. If T_1 has another distribution than other time intervals T_i, the point process is called a *general* or *delayed renewal process.* In that case the time origin is placed arbitrarily.

An ordinary renewal process can be defined equivalently as the sequence of positive, independent and identically distributed random variables $\{T_i, i = 1, 2, \dots\}$.

Consider an interval $[0, t[$ of the time-axis. An *ordinary renewal density* $h_o(t)$, Cox [4.5], Cox and Isham [4.6] represents the probability that a random point (not necessarily the first) occurs in $[t, t + dt[$, given that a random point occurs at the origin. A *modified renewal density* $h_m(t)$ represents the probability that a random point (not necessarily the first) occurs in $[t, t + dt[$, with arbitrarily chosen time-origin. A modified renewal density is the first-order product density of the renewal point process

$$h_m(t)dt = \Pr\{dN(t) = 1\} = f_1(t)dt \tag{4.27}$$

If this probability is irrespective of the position of the interval $[t, t+dt[$ on the time-axis, the renewal process is stationary.

Product densities of higher degrees of a renewal process appear to split into a suitable product form, Srinivasan [4.1]. To demonstrate that let us evaluate the joint probability that one point occurs in $[t_1, t_1 + dt_1[$ and one in $[t_2, t_2 + dt_2[$. This probability may be represented as

$$\begin{aligned}&\Pr\{N(t_1, t_1 + dt_1) = 1 \wedge N(t_2, t_2 + dt_2) = 1\} = \\ &\Pr\{N(t_2, t_2 + dt_2) = 1 \big| N(t_1, t_1 + dt_1) = 1\} \Pr\{N(t_1, t_1 + dt_1) = 1\}\end{aligned} \tag{4.28}$$

Equivalently, the probabilities of occurrence of points in the infinitesimal time intervals $[t_1, t_1 + dt_1[$ and $[t_2, t_2 + dt_2[$ can be expressed, respectively, as:

$$\Pr\{N(t_1, t_1 + dt_1) = 1\} = \Pr\left\{\left(\sum_{i=1}^{N(t_1)+1} T_i\right) \in [t_1, t_1 + dt_1[\right\} \tag{4.29}$$

$$\Pr\{N(t_2, t_2 + dt_2) = 1\} = \Pr\left\{\left(\sum_{i=1}^{N(t_2)+1} T_i\right) \in [t_2, t_2 + dt_2[\right\} \tag{4.30}$$

hence

$$\begin{aligned}&\Pr\{N(t_1, t_1 + dt_1) = 1 \wedge N(t_2, t_2 + dt_2) = 1\} = \\ &\Pr\left\{\left(\sum_{i=1}^{N(t_1)+1} T_i\right) \in [t_1, t_1 + dt_1[\bigwedge \left(\sum_{i=N(t_1)+2}^{N(t_2)+1} T_i\right) \in [t_2 - t_1, t_2 + dt_2 - t_1[\right\} = \\ &\Pr\left\{\left(\sum_{i=1}^{N(t_1)+1} T_i\right) \in [t_1, t_1 + dt_1[\right\} \Pr\left\{\left(\sum_{i=N(t_1)+2}^{N(t_2)+1} T_i\right) \in [t_2 - t_1, t_2 + dt_2 - t_1[\right\}\end{aligned} \tag{4.31}$$

The splitting of probability in (4.31) is due to the independence of random variables T_i. Then comparing (4.28) and (4.31) one can note that

$$\Pr\Big\{N(t_2,t_2+dt_2)=1\Big|N(t_1,t_1+dt_1)=1\Big\}=$$

$$\Pr\left\{\left(\sum_{i=N(t_1)+2}^{N(t_2)+1}T_i\right)\in[t_2-t_1,t_2+dt_2-t_1[\right\}=\Pr\left\{\left(\sum_{i=1}^{N(t_1,t_2)}T_i\right)\in[t_2-t_1,t_2+dt_2-t_1[\right\}=$$

$$\sum_{k=1}^{\infty}\Pr\left\{\left(\sum_{i=1}^{k}T_i\right)\in[t_2-t_1,t_2+dt_2-t_1[\right\} \tag{4.32}$$

Each of the probabilities that either one random variable T_i or the sum of two, or the sum of three random variables and so on, assumes value in the interval $t_2+dt_2-t_1$ is merely a function of t_2-t_1. It is so in the case of one random variable and also in the case of the sum of independent random variables, in which case this probability is obtained from the pertinent, multifold convolution of the individual density functions, the upper integration limit being t_2-t_1. Consequently the conditional probability in (4.28) being the probability of occurrence of one point in $[t_2,t_2+dt_2[$ given that one occurs in $[t_1,t_1+dt_1[$ is just the function of the difference t_2-t_1 and becomes expressed by the ordinary renewal density of the renewal process with the origin shifted by t_1, i.e.

$$\Pr\{N(t_2,t_2+dt_2)=1\Big|N(t_1,t_1+dt_1)=1\}=h_o(t_2-t_1)dt_2 \tag{4.33}$$

Seeing that

$$\Pr\{N(t_1,t_1+dt_1)=1\}=h_m(t_1)dt_1 \tag{4.34}$$

one obtains finally

$$\Pr\{N(t_1,t_1+dt_1)=1\wedge N(t_2,t_2+dt_2)=1\}=f_2(t_1,t_2)dt_1dt_2=$$

$$h_m(t_1)h_o(t_2-t_1)dt_1dt_2 \tag{4.35}$$

Proceeding likewise one obtains in general

$$f_n(t_1,\ldots,t_n)dt_1\cdots dt_n=E\big[dN(t_1)\cdots dN(t_n)\big]=$$

$$h_m(t_1)h_o(t_2-t_1)h_o(t_3-t_2)\cdots h_o(t_n-t_{n-1})dt_1dt_2\cdots dt_n \quad , \quad t_1<t_2<\ldots<t_n \tag{4.36}$$

hence

$$f_n(t_1,\ldots,t_n)=h_m(t_1)h_o(t_2-t_1)h_o(t_3-t_2)\cdots h_o(t_n-t_{n-1}) \quad , \quad t_1<t_2<\ldots<t_n \tag{4.37}$$

Let us denote the probability density of the random variable X_1 as $g_1(t)$ and the probability density of each of the variables $\{T_i,\ i = 2, 3, \dots\}$ as $g(t)$. It can be shown that the renewal densities $h_m(t)$ and $h_o(t)$ satisfy, respectively, inhomogeneous Volterra integral equations of the second kind, called *renewal equations*. These equations are derived, Srinivasan [4.1], by considering the fact that the occurrence of the point in $[t, t + dt[$ is due to two mutually exclusive events: either it is the first point, or it is the subsequent point. If it is the first point, the probability of its occurrence is just $g_1(t)dt$ (in the case of a delayed renewal process), or $g(t)dt$ (in the case of an ordinary renewal process). If it is the subsequent point, the preceding one has occurred at an arbitrary $t-u, u \in [0, t[$, u being the time interval between those two points. This leads to the following integral equations

$$h_m(t) = g_1(t) + \int_0^t h_m(t-u)g(u)du \tag{4.38}$$

$$h_o(t) = g(t) + \int_0^t h_o(t-u)g(u)du \tag{4.39}$$

The renewal densities can be evaluated by taking the Laplace transforms of the equations (4.38) and (4.39), which finally yields, Cox [4.5], Cox and Isham [4.6]

$$h_m(t) = \mathcal{L}^{-1}\left\{\frac{g_1^*(s)}{1-g^*(s)}\right\} \tag{4.40}$$

$$h_o(t) = \mathcal{L}^{-1}\left\{\frac{g^*(s)}{1-g^*(s)}\right\} \tag{4.41}$$

where $\mathcal{L}^{-1}\{\cdot\}$ denotes an inverse Laplace transform.

A class of the renewal processes which is important in applications are Erlang processes, where the time intervals between events have gamma (or Pearson type III) probability distribution, $X_i \sim \mathrm{G}(k-1, \nu)$, with the density function

$$g(t) = \frac{\nu^k}{(k-1)!}t^{k-1}\exp(-\nu t) \quad , \quad t > 0 \tag{4.42}$$

where $k = 1, 2, 3, \dots$ A Poisson process is a special case of such a process, specified by letting $k = 1$, in which case the time-intervals have the negative exponential distribution characterized by the density function

$$g(t) = \nu\exp(-\nu t) \quad , \quad t > 0 \tag{4.43}$$

An important property of the gamma distribution with density function given by (4.41) is that it is the distribution of the sum of k independent random variables, each of whose

distribution is negative-exponential with parameter ν. Hence the events driven by an Erlang process with parameter k can be regarded as every kth events of the generating Poisson process with the mean arrival rate ν.

The renewal densities of the Erlang process are (see Example Problem 4.1),

$$h_o(t) = \frac{\nu}{2}\Big(1 - \exp(-2\nu t)\Big) \quad , \quad k = 2 \tag{4.44}$$

$$h_o(t) = \frac{\nu}{3}\left[1 - \left(\sqrt{3}\sin\frac{\sqrt{3}}{2}\nu t + \cos\frac{\sqrt{3}}{2}\nu t\right)\exp\left(-\frac{3}{2}\nu t\right)\right] \quad , \quad k = 3 \tag{4.45}$$

$$h_o(t) = \frac{\nu}{4}\Big[1 - 2\sin(\nu t)\exp(-\nu t) - \exp(-2\nu t)\Big] \quad , \quad k = 4 \tag{4.46}$$

It is interesting to note that, although the Erlang events are every kth Poisson events, the renewal densities, which are the mean arrival rates of Erlang events, only asymptotically (as $t \to \infty$) tend to ν/k.

4.2 Random trains of overlapping pulses: filtered stochastic point processes

4.2.1 General case

A filtered stochastic point process $\{X(t), t \in [0, t[\}$ is defined as

$$X(t) = \sum_{i=1}^{N(t)} w\,(t, t_i, \mathbf{P}_i) \tag{4.47}$$

where $\{N(t), t \in [0, \infty[\}$ is a general counting process, $\mathbf{P}_i$ is a vector random variable attributed to a random point t_i. In a general case the random variables which are the components of the vector $\mathbf{P}_i$ do not have to be independent, they can be correlated, and can be characterized by different probability distributions. Nor the vector random variables $\mathbf{P}_i$ attributed to different points have to be mutually independent or identically distributed. The only assumption made at present is that these random variables are statistically independent of the counting process $\{N(t)\}$. The non-random function $w\,(t, t_i, \cdot)$, called the *filter function* represents the effect at the time t of an event occurring at the random instant t_i, the event being characterized by a vector random variable $\mathbf{P}_i$. For causality reasons we shall assume that $w\,(t, \tau, \mathbf{P}_i) = 0$ for $\tau > t$. Hence the process (4.47) represents the cumulative effect of a train of point events occurring at random instants t_i belonging to the interval $[0, t[$, described by a general stochastic point process. The process $X(t), t \in [0, \infty[\}$ defined by the formula (4.46) can be interpreted as a random train of general pulses, or signals, with origins at the random times t_i, $w\,(t, t_i, \mathbf{P}_i)$ being the *pulse shape function.*

After the division of the interval $[0, t[$ onto disjoint, contiguous subintervals, $X(t)$ can be written down as the Riemann-Stieltjes sum. The limit, in the mean-square sense, of the sequence of such sums, is the mean-square Riemann-Stieltjes integral with respect to the counting process $N(t)$, or the stochastic integral

$$X(t) = \int_0^t w\big(t, \tau, \mathbf{P}(\tau)\big) dN(\tau) \tag{4.48}$$

where $\mathbf{P}(\tau)$ is the vector random variable prescribed to the point occurring in the interval $[\tau, \tau + d\tau[$.

The expected value of the process $X(t)$ is obtained just by averaging the expression (4.47), which yields

$$E\big[X(t)\big] = \int_0^t E\Big[w\big(t, \tau, \mathbf{P}(\tau)\big)\Big] f_1(\tau) d\tau = \int_0^t \int_{\mathcal{P}_\tau} w(t, \tau, \mathbf{p}) f_{\mathbf{P}}(\mathbf{p}, \tau) f_1(\tau) d\mathbf{p} d\tau \tag{4.49}$$

where $f_{\mathbf{P}}(\mathbf{p}, \tau)$ is the joint probability density of the vector random variable $\mathbf{P}(\tau)$, which may be time-variant, and $\mathcal{P}_\tau$ is the sample space of this vector random variable.

Subsequent moments of the process $X(t)$ are evaluated by averaging of the pertinent multi-fold integrals obtained based on (4.48).

For example, the second-order moment (the mean square) is formulated as

$$E\big[X^2(t)\big] = \int_0^t \int_0^t E\Big[w\big(t, \tau_1, \mathbf{P}(\tau_1)\big) w\big(t, \tau_2, \mathbf{P}(\tau_2)\big)\Big] E\big[dN(\tau_1) dN(\tau_2)\big] \tag{4.50}$$

In order to evaluate this integral the degeneracy property of the second-degree product density must be taken into account, which takes place within the integration domain, for $\tau_1 = \tau_2$. This yields

$$E\big[X^2(t)\big] = \int_0^t E\Big[w^2\big(t, \tau, \mathbf{P}_\tau\big)\Big] f_1(\tau) d\tau +$$

$$\int_0^t \int_0^t E\Big[w\big(t, \tau_1, \mathbf{P}(\tau_1)\big) w\big(t, \tau_2, \mathbf{P}(\tau_2)\big)\Big] f_2(\tau_1, \tau_2) d\tau_1 d\tau_2 \tag{4.51}$$

Likewise the correlation function of the process $X(t)$ can be obtained as

$$E\big[X(t_1) X(t_2)\big] = \int_0^{\min(t_1, t_2)} E\Big[w\big(t_1, \tau, \mathbf{P}(\tau)\big) w\big(t_2, \tau, \mathbf{P}(\tau)\big)\Big] f_1(\tau) d\tau +$$

$$\int_0^{t_1}\int_0^{t_2} E\Big[w\big(t_1,\tau_1,\mathbf{P}(\tau_1)\big)w\big(t_2,\tau_2,\mathbf{P}(\tau_2)\big)\Big] f_2(\tau_1,\tau_2)d\tau_1 d\tau_2 \tag{4.52}$$

The general expression for the nth order moment is

$$E\big[X^n(t)\big] = \underbrace{\int_0^t \cdots \int_0^t}_{n\text{–fold}} E\left[\prod_{k=1}^{n} w\big(t,\tau_k,\mathbf{P}(\tau_k)\big)\right] E\left[\prod_{k=1}^{n} dN\left(\tau_k\right)\right] \tag{4.53}$$

Of course, as $\tau_1 \neq \tau_2 \neq \ldots \neq \tau_n$, then

$$E\left[\prod_{k=1}^{n} dN(\tau_k)\right] = f_n(\tau_1,\ldots,\tau_n)d\tau_1\cdots d\tau_n \tag{4.54}$$

In the multidimensional integration domain any possible equatings of the arguments τ_k take place. Therefore, in order to evaluate the integral (4.53) all possible degeneracies of nth degree product density $f_n\left(\tau_1,\ldots,\tau_n\right)$ must be taken into account. Moreover, in general case the integration is performed with respect to the joint probability density of n vector random variables $\mathbf{P}_k, k = 1,2,\ldots,n$.

In particular the expression for the third-order moment is obtained as

$$E\big[X^3(t)\big] = \int_0^t E\left[w^3\left(t,\tau,\mathbf{P}(\tau)\right)\right] f_1(\tau)d\tau +$$

$$3\int_0^t\int_0^t E\left[w^2(t,\tau_1,\mathbf{P}(\tau_1))w\big(t,\tau_2,\mathbf{P}(\tau_2)\big)\right] f_2(\tau_1,\tau_2)d\tau_1 d\tau_2 +$$

$$\int_0^t\int_0^t\int_0^t E\left[w\big(t,\tau_1,\mathbf{P}(\tau_1)\big)w\big(t,\tau_2,\mathbf{P}(\tau_2)\big)w\big(t,\tau_3,\mathbf{P}(\tau_3)\big)\right] f_3(\tau_1,\tau_2,\tau_3)d\tau_1 d\tau_2 d\tau_3 \tag{4.55}$$

However, in general, the evaluation of the above integrals becomes cumbersome, especially in the case of higher-order moments. Then, in the case of a Poisson process, it is much easier to handle the cumulants, which can be obtained directly from the log-characteristic function, called also a cumulant generating function. In the case of a filtered renewal process the recursive expressions for the moments can be obtained from the integral equations governing the characteristic function.

4.2.2 Filtered Poisson process

A filtered Poisson process $\{X(t), t \in [0, \infty[\}$ is defined as

$$X(t) = \sum_{i=1}^{N(t)} w\,(t, t_i, P_i) \tag{4.56}$$

where $\{N(t), t \in [0, \infty[\}$ is the counting Poisson process and P_i are independent, identically distributed random variables having the distribution as a common random variable P. The random variables P_i are also assumed to be independent of the counting process $N(t)$.

The characteristic function of a filtered Poisson process (4.56) defined as

$$\Phi_X(\theta, t) = E\big[\exp\{i\theta X(t)\}\big] = E\left[\exp\left(i\theta \sum_{i=1}^{N(t)} w(t, t_i, P_i)\right)\right] \tag{4.57}$$

may be expressed with the help of the integral representation (4.48) as

$$\Phi_X(\theta, t) = E\left[\exp\left(i\theta \int_0^t w\big(t, \tau, P(\tau)\big) dN(\tau)\right)\right] \tag{4.58}$$

Using auxiliary division of the interval $[0, t[$ into m contiguous subintervals, of length $\Delta\tau_k$ each, i.e. $[\tau_k, \tau_k + \Delta\tau_k[$ and representing the stochastic integral in (4.58) as the mean square limit of the sequence of the Riemann-Stieltjes sums, one obtains, cf. section 1.1.2

$$\Phi_X(\theta, t) = \lim_{\Delta\tau_{\max} \to 0} E\left[\exp\left(i\theta \sum_{k=1}^{m} w\big(t, \tau_k, P(\tau_k)\big) \Delta N(\tau_k)\right)\right] \tag{4.59}$$

where $\Delta N(\tau_k) = N(\tau_k + \Delta\tau_k) - N(\tau_k)$.

Next upon factorizing the exponential of the sum and splitting the expectation one has

$$\Phi_X(\theta, t) = \lim_{\Delta\tau_{\max} \to 0} E\left[\prod_{k=1}^{m} \exp\big(i\theta w\big(t, \tau_k, P(\tau_k)\big) \Delta N(\tau_k)\big)\right] =$$

$$\lim_{\Delta\tau_{\max} \to 0} \prod_{k=1}^{m} E\big[\exp\big(i\theta w\big(t, \tau_k, P(\tau_k)\big) \Delta N(\tau_k)\big)\big] \tag{4.60}$$

The expectation is performed as

$$E\Big[\exp\Big(i\theta w(t,\tau_k,P(\tau_k))\Delta N(\tau_k)\Big)\Big] =$$

$$E\Big[\exp\Big(i\theta w(t,\tau_k,P(\tau_k))\Delta N(\tau_k)\Big)\Big|\Delta N(\tau_k)=0\Big]\Pr\{\Delta N(\tau_k)=0\}+$$

$$E\Big[\exp\Big(i\theta w(t,\tau_k,P(\tau_k))\Delta N(\tau_k)\Big)\Big|\Delta N(\tau_k)=1\Big]\Pr\{\Delta N(\tau_k)=1\} =$$

$$1\cdot\Big(1-\nu(\tau_k)\Delta\tau_k+E\Big[\exp\Big(i\theta w(t,\tau_k,P(\tau_k))\Big)\Big]\nu(\tau_k)\Delta\tau_k =$$

$$1+\Big(E\Big[\exp\Big(i\theta w(t,\tau_k,P(\tau_k))\Big)\Big]-1\Big)\nu(\tau_k)\Delta\tau_k \tag{4.61}$$

Taking the logarithms of both sides of (1.59) and using the MacLaurin expansion $\ln(1+x)=x+\mathrm{O}(x^2)$ yields

$$\ln\Phi_X(\theta,t)=\lim_{\Delta\tau_{\max}\to 0}\sum_{k=1}^{m}\ln\left(1+\left(E\left[\exp\Big(i\theta w(t,\tau_k,P(\tau_k))\Big)\right]-1\right)\nu(\tau_k)\Delta\tau_k\right)=$$

$$\lim_{\Delta\tau_{\max}\to 0}\sum_{k=1}^{m}\left(E\left[\exp\Big(i\theta w(t,\tau_k,P(\tau_k))\Big)\right]-1\right)\nu(\tau_k)\Delta\tau_k \tag{4.62}$$

Hence, cf. Parzen [4.6]

$$\Phi_X(\theta,t)=\exp\left\{\int_0^t E\Big[\exp\big(i\theta w(t,\tau,P)\big)-1\Big]\nu(\tau)d\tau\right\} \tag{4.63}$$

The cumulants $\lambda_n(t)$ are obtained by differentiation of the log-characteristic function $\ln\Phi_X(\theta,t)$ as

$$\lambda_n(t)=i^{-n}\frac{d^n\ln\Phi_X(\theta,t)}{d\theta^n}\Big|_{\theta=0} \tag{4.64}$$

which yields

$$\lambda_n(t)=\int_0^t E\Big[w^n(t,\tau,P)\Big]\nu(\tau)d\tau \tag{4.65}$$

where the mean value $\mu(t)=E[X(t)]=\lambda_1(t)$ and the variance $\sigma^2(t)=\lambda_2(t)$.

If the pulse shape function is of the separable form i.e. $w(t, t_i, P_i) = P_i w(t, t_i)$, then

$$\Phi_X(\theta, t) = \exp\left\{ \int_0^t \Big[\Phi_P\big(i\theta w(t,\tau)\big) - 1 \Big] \nu(\tau) d\tau \right\} \tag{4.66}$$

where $\Phi_P(\cdot)$ is the characteristic function of the random variable P. Then the expression (4.65) for the cumulants simplifies to the form of

$$\lambda_n(t) = E[P^n] \int_0^t w^n(t,\tau)\nu(\tau)d\tau \tag{4.67}$$

The expressions (4.63) and (4.65) or (4.66) and (4.67) can easily be generalized to the case of a random vector $\mathbf{P}$.

4.2.3 Filtered renewal process

Consider a filtered process $\{X(t), t \in [0, \infty[\}$, driven by an ordinary renewal counting process $\{X(t), t \in [0, \infty[\}$ in the form of

$$X(t) = \sum_{j=1}^{N(t)} w\big(t - t_j, P_j\big) \tag{4.68}$$

where the filter function is assumed to be causal, i.e. $w(\tau, P_j) = 0$ for $\tau < 0$. The random variables P_j are assumed to be independent and to have identical probability distributions characterized by the common density function $f_P(p)$. The probability distributions of the interarrival times are characterized by the probability distribution function $G(t)$ and by the probability density function $g(t)$.

The characteristic function $\Phi_X(\theta, t)$ of the filtered renewal process $X(t)$ defined by (4.67) appears to satisfy the inhomogeneous Volterra integral equation of the second kind. The following derivation is due to Takacs [4.7].

The filtered process $X(t)$ given by (4.68) may be regarded as a sum of the first pulse occurring after first interarrival time T_1 and the filtered process with the origin shifted by T_1, i.e. $X(t - T_1)$; thus

$$X(t) = w(t - T_1, P_1) + X(t - T_1) \tag{4.69}$$

Suppose that $T_1 = \tau$. In view of independence of random variables P_j, the conditional characteristic function, given that the first pulse occurs at the time $T_1 = \tau$ is expressed as

$$\Phi_X(\theta, t \Big| T_1 = \tau) = E\Big[\exp\big(i\theta X(t)\Big| T_1 = \tau\big)\Big] =$$

$$E\Big[\exp\big(iw(t-\tau, P_1)\big)\exp\big(i\theta X(t-\tau)\big)\Big] = \Gamma(\theta, t-\tau)\Phi_X(\theta, t-\tau) \tag{4.70}$$

where

$$\Gamma(\theta, t-\tau) = E\Big[\exp\{i\theta w(t-\tau, P)\}\Big] = \int_{-\infty}^{\infty} \exp\big[i\theta w(t-\tau, p)\big] f_P(p) dp \tag{4.71}$$

is the characteristic function of a single general pulse $w(t-\tau, P)$. By unconditioning one obtains

$$\Phi_X(\theta, t) = \int_0^t \Phi_X(\theta, t-\tau)\Gamma(\theta, t-\tau) g(\tau) d\tau + C \tag{4.72}$$

where the integration constant C is evaluated from the obvious condition for $\theta = 0$

$$\Phi_X(0, t) = 1 = \int_0^t g(\tau) d\tau + C \tag{4.73}$$

thus $C = 1 - G(t)$. Hence the final result is

$$\Phi_X(\theta, t) = \int_0^t \Phi_X(\theta, t-\tau)\Gamma(\theta, t-\tau) g(\tau) d\tau + 1 - G(t) \tag{4.74}$$

After differentiating the equation (4.74) r times with respect to θ and after substituting $\theta = 0$ the following integral equation governing the moment $E\left[X^r(t)\right]$ is arrived at

$$E\big[X^r(t)\big] = \sum_{i=0}^{r} \binom{r}{i} \int_0^t \phi_{r-i}(t-\tau) E\big[X^i(t-\tau)\big] g(\tau) d\tau \tag{4.75}$$

where

$$\phi_k(t-\tau) = E\big[w^k(t-\tau, P)\big] = \int_{-\infty}^{\infty} w^k(t-\tau, p) f_P(p) dp \tag{4.76}$$

Taking the Laplace transforms of the both sides of equation (4.75), solving the obtained algebraic equation for the transform of the rth order moment and taking the inverse transform one obtains the following recursive expression, Takacs [4.7].

$$E\big[X^r(t)\big] = \sum_{i=0}^{r-1} \binom{r}{i} \int_0^t \phi_{r-i}(t-\tau) E\big[X^i(t-\tau)\big] h_o(\tau) d\tau \tag{4.77}$$

where $h_o(\tau)$ is the ordinary renewal density, see (4.41).

The specific formulae for the mean value function $E[X(t)]$ and the mean square function $E[X^2(t)]$ are obtained as

$$E[X(t)] = \int_0^t \phi_1(t-\tau)h_o(\tau)d\tau \tag{4.78}$$

$$E[X^2(t)] = \int_0^t \phi_2(t-\tau)h_o(\tau)d\tau + 2\int_0^t \int_\tau^t \phi_1(t-\tau)\phi_1(t-u)h_o(u-\tau)h_o(\tau)dud\tau \tag{4.79}$$

where

$$\phi_1(t-\tau) = E[w(t-\tau, P)] \tag{4.80}$$

$$\phi_2(t-\tau) = E[w^2(t-\tau, P)] \tag{4.81}$$

4.3 Random trains of non-overlapping pulses with Erlang arrivals and truncated Erlang durations: queueing theory methods

4.3.1 Statement of the problem for rectangular pulses

Consider a train of rectangular pulses whose arrival times are distributed according to an Erlang renewal process, i.e. the interarrival times T_a are gamma-distributed random variables. The pulses are assumed not to overlap, i.e. each pulse duration completes before, or is truncated at, the moment of the next pulse arrival. This means that the actual duration T_d of the pulse, equals the duration T'_d sampled from the original (or primitive) gamma (Erlang) distribution if the duration completes before the next arrival, and the actual duration is equal the interarrival time if it has not completed until the next arrival. In the latter case two consecutive pulses adjoin each other. This situation, under the assumption that at the initial time no pulse is acting, is shown in fig. 4.1.

Hence the truncated pulse duration T_d is defined as

$$T_d = \begin{cases} T'_d & , \text{ if } T'_d < T_a \\ T_a & , \text{ if } T'_d \geq T_a \end{cases} \tag{4.82}$$

The pulses magnitudes, or heights, are given by random variables P_i, which are the mark variables associated with the arrival times t_i. These variables are assumed to be independent and identically distributed as a random variable P.

The process $X(t)$ as defined above may be expressed as

$$X(t) = \begin{cases} P_i & , \text{ pulse "on" after arrival} \\ 0 & , \text{ pulse "off"} \end{cases} \tag{4.83}$$

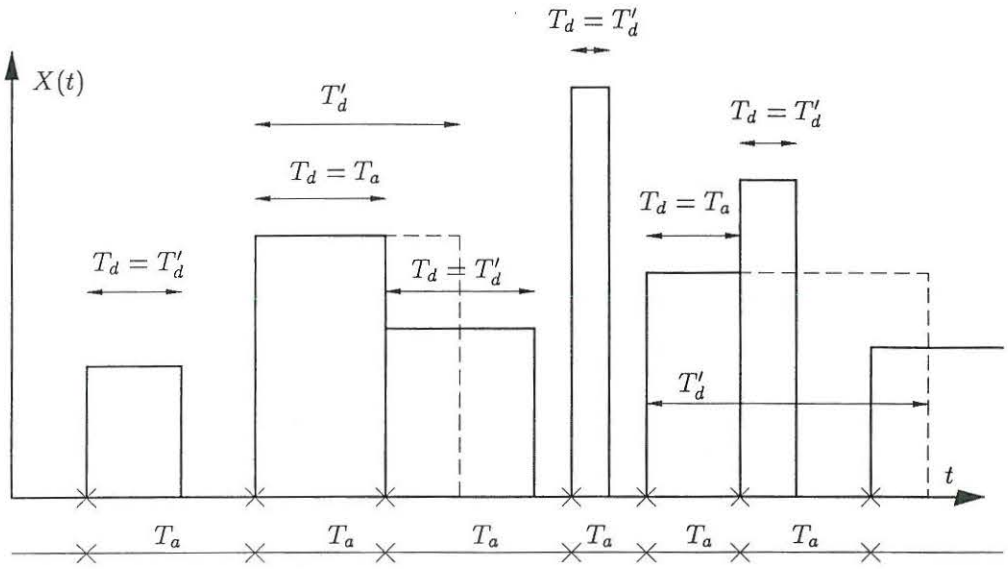

Fig. 4.1. Train of non-overlapping, truncated rectangular pulses.

It is here convenient to represent the probability density of interarrival times T_a in the form of

$$g_{T_a}(t) = \frac{(\nu k)^k}{(k-1)!} t^{k-1} \exp(-\nu k t) \quad , \quad t > 0 \tag{4.84}$$

As seen from (4.42), this defines the mean arrival rate of generating Poisson events as νk. The primitive pulse duration T_d' has the probability density

$$g_{T_d'}(t) = \frac{(\mu l)^l}{(l-1)!} t^{l-1} \exp(-\mu l t) \quad , \quad t > 0 \tag{4.85}$$

The probability density function $g_{T_d}(t)$ of the truncated pulse duration defined by (4.81) is hence expressed as

$$\begin{aligned} & g_{T_d}(t)dt = \Pr\{T_d \in (t, t+dt)\} = \\ & \Pr\{T_d \in [t, t+dt[\mid T_d' < T_a\} \Pr\{T_d' < T_a\} + \\ & \Pr\{T_d \in [t, t+dt[\mid T_d' \geq T_a\} \Pr\{T_d' \geq T_a\} = \\ & \Pr\{T_d' \in [t, t+dt[\} \Pr\{T_a > t\} + \Pr\{T_a \in [t, t+dt[\} \Pr\{T_d' \geq t\} \end{aligned} \tag{4.86}$$

The result is

$$g_{T_d}(t) = g_{T_d'}(t)\big(1 - F_{T_a}(t)\big) + g_{T_a}(t)\big(1 - F_{T_d'}(t)\big) \tag{4.87}$$

The probability distribution function $F_{T_d}(t)$ is obtained as

$$F_{T_d}(t) = \int_0^t g_{T_d}(\tau)d\tau = \int_0^t g_{T'_d}(\tau)\big(1 - F_{T_a}(\tau)\big)d\tau + \int_0^t g_{T_a}(\tau)\big(1 - F_{T'_d}(\tau)\big)d\tau \quad (4.88)$$

Integrating by parts yields the result, Shinozuka [4.8].

$$F_{T_d}(t) = F_{T'_d}(\tau)\Big(1 - F_{T_a}(\tau)\Big)\Big|_0^t + \int_0^t F_{T'_d}(\tau)g_{T_a}(\tau)d\tau +$$

$$F_{T_a}(\tau)\Big(1 - F_{T'_d}(\tau)\Big)\Big|_0^t + \int_0^t F_{T_a}(\tau)g_{T'_d}(\tau)d\tau =$$

$$F_{T'_d}(t) - F_{T'_d}(t)F_{T_a}(t) + F_{T_a}(t) - F_{T_a}(t)F_{T'_d}(t) + F_{T_a}(t)F_{T'_d}(t) =$$

$$F_{T'_d}(t) + F_{T_a}(t) - F_{T'_d}(t)F_{T_a}(t) = 1 - \Big(1 - F_{T'_d}(t)\Big)\Big(1 - F_{T_a}(t)\Big) \quad (4.89)$$

The remaining "off" time between the consecutive pulses, i.e. the time gap between them $T_r = T_a - T_d$, $T_r > 0$, may be expressed as

$$T_r = \begin{cases} T_a - T'_d & , \quad T_a > T'_d \\ 0 & , \quad T_a \leq T'_d \end{cases} \quad (4.90)$$

The probability density function $g_{T_r}(t)$ of the remaining time T_r is expressed as

$$g_{T_r}(t)dt = \Pr\{T_r \in [t, t + dt[\} =$$

$$\Pr\{T_a - T_d \in [t, t + dt[\wedge T_a - T_d > 0\} + \Pr\{T_a - T_d \in [t, t + dt[\wedge T_a - T_d = 0\} =$$

$$\Pr\{T_a - T'_d \in [t, t + dt[\} + \Pr\{T_a - T_d \in [t, t + dt[|\ T_a - T_d = 0\} \Pr\{T_a - T_d = 0\} =$$

$$\Pr\{T_a \in [T'_d + t, T'_d + t + dt[\} + \Pr\{T_a \in [T_a + t, T_a + t + dt[\} \Pr\{T_a \leq T'_d\} \quad (4.91)$$

which yields

$$g_{T_r}(t) = \int_0^\infty g_{T_a}(t + \tau)g_{T'_d}(\tau)d\tau + \delta(t) \int_0^\infty F_{T_a}(\tau)g_{T'_d}(\tau)d\tau \quad (4.92)$$

The probability distribution function $F_{T_r}(t)$ is then obtained as

$$F_{T_r}(t) = \int_0^t g_{T_r}(\tau)d\tau = \int_0^\infty F_{T_a}(t + \tau)g_{T'_d}(\tau)d\tau + \mathbf{1}(t) \int_0^\infty F_{T_a}(\tau)g_{T'_d}(\tau)d\tau \quad (4.93)$$

where $\mathbf{1}(t)$ is the Heaviside's unit step function, cf. (1.18).

4.3.2 Differential equations governing the Markov state probabilities

Since the arrival and loading processes are Erlang distributed with integer parameters k and l, respectively, the so-called "phase approach" of the queueing theory will be used [4.9, 4.10]. The interarrival times and the loading processes (pulse durations) are made up of the negative exponential distributed variates, called "phases". The states of the process are defined as the coincidences of different phases of the arrival and loading processes. Due account must also be taken of the fact that when the pulse is truncated then two consecutive pulses are contiguous. Thus the arrivals of phases are Poisson distributed with parameter νk. Hence a new phase of the arrival process occurs in the time interval $[t, t + \Delta t[$ with probability $\nu k \Delta t$. A phase of the loading process is completed in the time interval $[t, t+\Delta t[$ with probability $\mu l \Delta t$. There may be k "empty" phases of no load and each of k phases of the arrival process may coincide with each of l phases of the loading process. Thus the total number of different states is $k(l+1)$. Let us enumerate the states as $i = 1, 2, \ldots, k(l+1)$. Different possible sequences of states, in the case $k = 2$ and $l = 2$, are illustrated in figure 4.2.

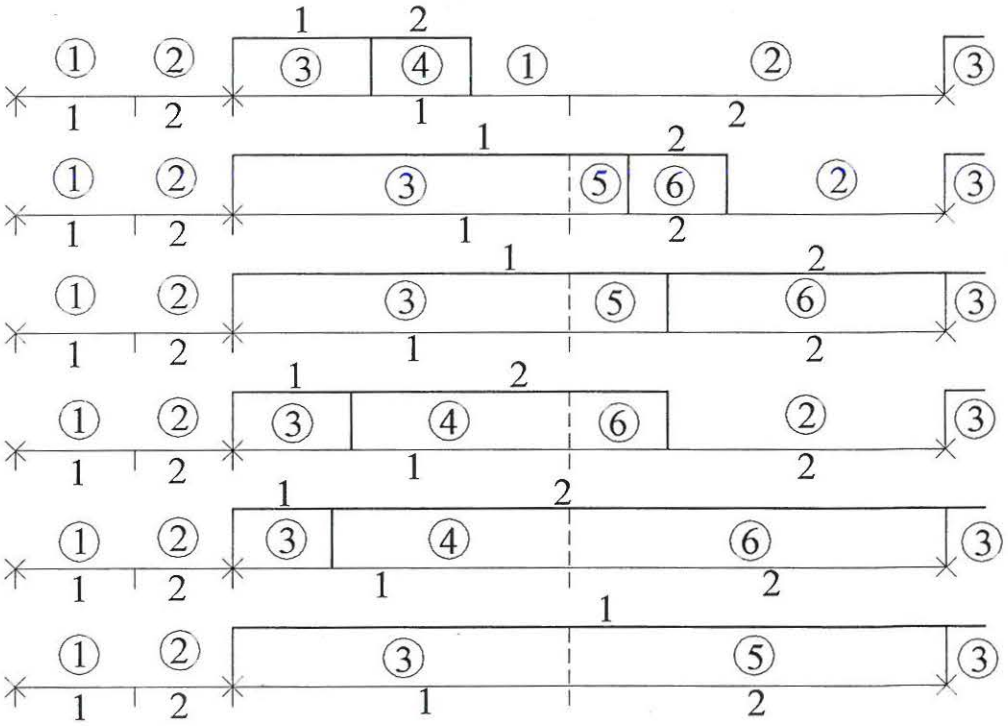

Fig. 4.2. Sequence of different possible states of a train of non-overlapping rectangular pulses with Erlang arrivals and truncated Erlang durations, in the case of $k = 2$ and $l = 2$.

These states are described by a Markov process $S(t)$, because the probability of the system being in a given state at the subsequent time depends merely on the states at the preceding time. The differential equations governing the probabilities $\mathcal{P}_i(t)$ that the system is in state i at time t, i.e. $\mathcal{P}_i(t) = \Pr\{S(t) = i\}$ are derived according to the general scheme

$$\mathcal{P}_i(t + \Delta t) = \sum_{j=0}^{k(l+1)} \Pr\{S(t + \Delta t) = i \mid S(t) = j\}\mathcal{P}_j(t) \tag{4.94}$$

where the conditional probabilities are just the transition probabilities of the pertinent Markov chain. An example chain of Markov states, for $k = 2$ and $l = 2$ is shown in fig. 4.3. In the present problem the following equations are obtained [4.11, 4.12]:

$$\left.\begin{aligned}
&\mathcal{P}_1(t + \Delta t) = \mathcal{P}_1(t)(1 - \nu k\Delta t) + \mathcal{P}_{k+l}(t)\mu l\Delta t \\
&\mathcal{P}_i(t + \Delta t) = \mathcal{P}_{i-1}(t)\nu k\Delta t + \mathcal{P}_i(t)(1 - \nu k\Delta t) + \mathcal{P}_{k+il}(t)\mu l\Delta t \ , \ 1 < i \leq k \\
&\mathcal{P}_{k+1}(t + \Delta t) = \mathcal{P}_k(t)\nu k\Delta t + \mathcal{P}_{k+1}(t)(1 - \nu k\Delta t)(1 - \mu l\Delta t) + \sum_{r=0}^{l-1} \mathcal{P}_{k(l+1)-r}(t)\nu k\Delta t \\
&\mathcal{P}_i(t + \Delta t) = \mathcal{P}_{i-1}(t)\mu l\Delta t + \mathcal{P}_i(t)(1 - \nu k\Delta t)(1 - \mu l\Delta t) \ , \ k + 1 < i \leq k + l \\
&\mathcal{P}_i(t + \Delta t) = \mathcal{P}_{i-l}(t)\nu k\Delta t + \mathcal{P}_{i-1}(t)\mu l\Delta t + \mathcal{P}_i(t)(1 - \nu k\Delta t)(1 - \mu l\Delta t) \\
&i > k + l \ , \ i \neq k + sl + 1 \ , \ s = 1, \ldots, k - 1 \\
&\mathcal{P}_i(t + \Delta t) = \mathcal{P}_{i-l}(t)\nu k\Delta t + \mathcal{P}_i(t)(1 - \nu k\Delta t)(1 - \mu l\Delta t) \\
&i > k + l \ , \ i = k + sl + 1 \ , \ s = 1, \ldots, k - 1
\end{aligned}\right\} \tag{4.95}$$

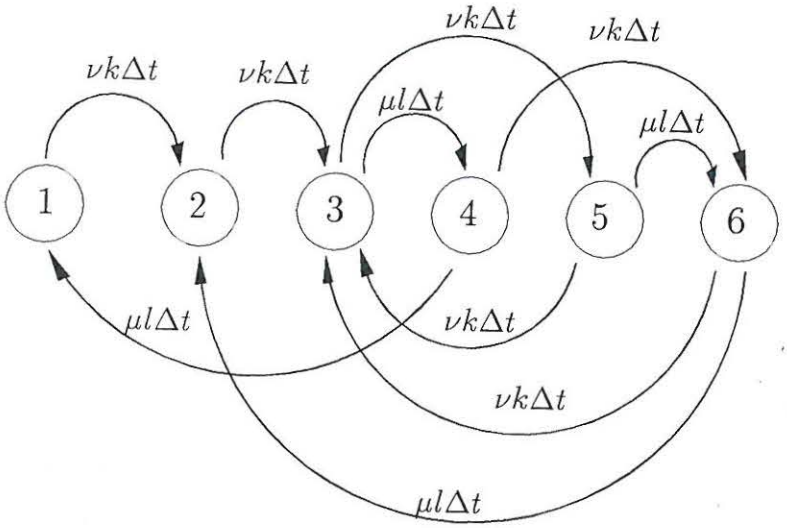

Fig. 4.3. Scheme of a chain of Markov states for a train of non-overlapping rectangular pulses with Erlang arrivals and truncated Erlang durations, in the case of $k = 2$ and $l = 2$.

The corresponding differential equations are written as, see Iwankiewicz and Rackwitz [4.11], [4.12]

$$\left.\begin{aligned}
&\dot{\mathcal{P}}_1 = -\mathcal{P}_1 \nu k + \mathcal{P}_{k+l} \mu l \\
&\dot{\mathcal{P}}_i = \mathcal{P}_{i-1} \nu k - \mathcal{P}_i \nu k + \mathcal{P}_{k+il} \mu l \quad , \quad 1 < i \leq k \\
&\dot{\mathcal{P}}_{k+1} = \mathcal{P}_k \nu k - \mathcal{P}_{k+1}(\nu k + \mu l) + \sum_{r=0}^{l-1} \mathcal{P}_{k(l+1)-r} \nu k \\
&\dot{\mathcal{P}}_i = \mathcal{P}_{i-1} \mu l - \mathcal{P}_i(\nu k + \mu l) \quad , \quad k+1 < i \leq k+l \\
&\dot{\mathcal{P}}_i = \mathcal{P}_{i-l} \nu k + \mathcal{P}_{i-1} \mu l - \mathcal{P}_i(\nu k + \mu l) \\
&i > k+l, \; i \neq k + sl + 1 \; , \; s = 1, \ldots, k-1 \\
&\dot{\mathcal{P}}_i = \mathcal{P}_{i-l} \nu k - \mathcal{P}_i(\nu k + \mu l) \; , \; i > k+l, \subset = k + sl + 1 \; , \; s = 1, \ldots, k-1
\end{aligned}\right\} \tag{4.96}$$

The probabilities $\mathcal{P}_i(t)$ must satisfy the normalization condition

$$\sum_{i=1}^{k(l+1)} \mathcal{P}_i(t) = 1 \tag{4.97}$$

The probabilities of the load being "off" and "on" are given, respectively, by

$$\mathcal{P}_{\text{off}}(t) = \Pr\left\{ \bigvee_{i=1}^{k} S(t) = i \right\} = \sum_{i=1}^{k} \mathcal{P}_i(t) \tag{4.98}$$

$$\mathcal{P}_{\text{on}}(t) = \Pr\left\{ \bigvee_{i=k+1}^{k(l+1)} S(t) = i \right\} = \sum_{i=k+1}^{k(l+1)} \mathcal{P}_i(t) \tag{4.99}$$

The associated initial conditions

$$\mathcal{P}_{0,i} = \mathcal{P}_i(0) = \Pr\{S(0) = i\} \; , \; i = 1, 2, \ldots, k(l+1) \tag{4.100}$$

may be specify that at the initial instant the pulse is "off"

$$\sum_{i=1}^{k} \mathcal{P}_i(0) = 1 \tag{4.101}$$

that it is "on"

$$\sum_{i=k+1}^{k(l+1)} \mathcal{P}_i(0) = 1 \tag{4.102}$$

or that the start is completely random

$$\sum_{i=1}^{k(l+1)} \mathcal{P}_i(0) = 1. \tag{4.103}$$

4.3.3 Moments of the train of rectangular non-overlapping pulses

Mean value of the process $X(t)$ defined by (4.82) and (4.83) is obtained as, cf. (4.98) and (4.99)

$$\mu_X(t) = E\left[X(t)\Big|\bigvee_{i=1}^{k} S(t)=i\right]\Pr\left\{\bigvee_{i=1}^{k} S(t)=i\right\}+$$
$$E\left[X(t)\Big|\bigvee_{i=k+1}^{k(l+1)} S(t)=i\right]\Pr\left\{\bigvee_{i=k+1}^{k(l+1)} S(t)=i\right\}=$$
$$0\cdot\mathcal{P}_{\text{off}}(t)+E[P]\mathcal{P}_{\text{on}}(t)=E[P]\mathcal{P}_{\text{on}}(t) \tag{4.104}$$

The auto-correlation function $\mu_{XX}(t_1,t_2)=E\Big[X(t_1)X(t_2)\Big]$, $t_1<t_2$, of the train of non-overlapping rectangular pulses $X(t)$ defined by (4.82) and (4.83) may be evaluated as

$$\mu_{XX}(t_1,t_2)=E\left[X(t_1)X(t_2)\Big|\left(\bigvee_{i=1}^{k} S(t_1)=i\right)\wedge\left(\bigvee_{j=1}^{k} S(t_2)=j\right)\right]\times$$
$$\Pr\left\{\left(\bigvee_{i=1}^{k} S(t_1)=i\right)\wedge\left(\bigvee_{j=1}^{k} S(t_2)=j\right)\right\}+$$
$$E\left[X(t_1)X(t_2)\Big|\left(\bigvee_{i=1}^{k} S(t_1)=i\right)\wedge\left(\bigvee_{j=k+1}^{k(l+1)} S(t_2)=j\right)\right]\times$$
$$\Pr\left\{\left(\bigvee_{i=1}^{k} S(t_1)=i\right)\wedge\left(\bigvee_{j=k+1}^{k(l+1)} S(t_2)=j\right)\right\}+$$
$$E\Big[X(t_1)X(t_2)\Big|\left(\bigvee_{i=k+1}^{k(l+1)} S(t_1)=i\right)\wedge\left(\bigvee_{j=1}^{k} S(t_2)=j\right)\Big]\times$$
$$\Pr\left\{\left(\bigvee_{i=k+1}^{k(l+1)} S(t_1)=i\right)\wedge\left(\bigvee_{j=1}^{k} S(t_2)=j\right)\right\}+$$
$$E\left[X(t_1)X(t_2)\Big|\left(\bigvee_{i=k+1}^{k(l+1)} S(t_1)=i\right)\wedge\left(\bigvee_{j=k+1}^{k(l+1)} S(t_2)=j\right)\right]\times$$
$$\Pr\left\{\left(\bigvee_{i=k+1}^{k(l+1)} S(t_1)=i\right)\wedge\left(\bigvee_{j=k+1}^{k(l+1)} S(t_2)=j\right)\right\}=$$
$$E\left[X(t_1)X(t_2)\Big|\left(\bigvee_{i=k+1}^{k(l+1)} S(t_1)=i\right)\wedge\left(\bigvee_{j=k+1}^{k(l+1)} S(t_2)=j\right)\right]\times$$
$$\Pr\left\{\left(\bigvee_{i=k+1}^{k(l+1)} S(t_1)=i\right)\wedge\left(\bigvee_{j=k+1}^{k(l+1)} S(t_2)=j\right)\right\} \tag{4.105}$$

The result is justified by the fact that all other expectations are equal zero. The probability $\Pr\left\{\left(\bigvee_{i=k+1}^{k(l+1)} S(t_1)=i\right)\wedge\left(\bigvee_{j=k+1}^{k(l+1)} S(t_2)=j\right)\right\}$ is just the probability that at the time t_1 the pulse is "on", i.e. $X(t_1)=P(t_1)$ and at the time t_2 the pulse is "on", i.e. $X(t_2)=P(t_2)$. This situation can arise in two exhaustive and mutually exclusive ways: either it is the same pulse and $P(t_1)=P(t_2)$ or these are two different pulses and $P(t_1)\neq P(t_2)$. Consequently the auto-correlation function may be expressed as

$$\mu_{XX}(t_1,t_2)=E\left[X(t_1)X(t_2)\middle|\left(\bigvee_{i=k+1}^{k(l+1)} S(t_1)=i\right)\wedge\left(\bigvee_{j=k+1}^{k(l+1)} S(t_2)=j\right)\wedge X(t_1)=X(t_2)\right]\times$$

$$\Pr\left\{\left(\bigvee_{i=k+1}^{k(l+1)} S(t_1)=i\right)\wedge\left(\bigvee_{j=k+1}^{k(l+1)} S(t_2)=j\right)\wedge X(t_1)=X(t_2)\right\}+$$

$$E\left[X(t_1)X(t_2)\middle|\left(\bigvee_{i=k+1}^{k(l+1)} S(t_1)=i\right)\wedge\left(\bigvee_{j=k+1}^{k(l+1)} S(t_2)=j\right)\wedge X(t_1)\neq X(t_2)\right]\times$$

$$\Pr\left\{\left(\bigvee_{i=k+1}^{k(l+1)} S(t_1)=i\right)\wedge\left(\bigvee_{j=k+1}^{k(l+1)} S(t_2)=j\right)\wedge X(t_1)\neq X(t_2)\right\}=$$

$$E\left[P^2\right]\times\Pr\left\{\left(\bigvee_{i=k+1}^{k(l+1)} S(t_1)=i\right)\wedge\left(\bigvee_{j=k+1}^{k(l+1)} S(t_2)=j\right)\wedge P(t_1)=P(t_2)\right\}+$$

$$E^2[P]\times\Pr\left\{\left(\bigvee_{i=k+1}^{k(l+1)} S(t_1)=i\right)\wedge\left(\bigvee_{j=k+1}^{k(l+1)} S(t_2)=j\right)\wedge P(t_1)\neq P(t_2)\right\}=$$

$$E\left[P^2\right]\times\left(\Pr\left\{\left(\bigvee_{i=k+1}^{j} S(t_1)=i\right)\wedge\left(\bigvee_{j=k+1}^{k(l+1)} S(t_2)=j\right)\wedge P(t_1)=P(t_2)\right\}+\right.$$

$$\left.\Pr\left\{\left(\bigvee_{i=j+1}^{k(l+1)} S(t_1)=i\right)\wedge\left(\bigvee_{j=k+1}^{k(l+1)} S(t_2)=j\right)\wedge P(t_1)=P(t_2)\right\}\right)+$$

$$E^2[P]\times\left(\Pr\left\{\left(\bigvee_{i=k+1}^{j} S(t_1)=i\right)\wedge\left(\bigvee_{j=k+1}^{k(l+1)} S(t_2)=j\right)\wedge P(t_1)\neq P(t_2)\right\}+\right.$$

$$\left.\Pr\left\{\left(\bigvee_{i=j+1}^{k(l+1)} S(t_1)=i\right)\wedge\left(\bigvee_{j=k+1}^{k(l+1)} S(t_2)=j\right)\wedge P(t_1)\neq P(t_2)\right\}\right) \tag{4.106}$$

Let us notice that for $\left(\bigvee_{i=j+1}^{k(l+1)} S(t_1)=i\right)\wedge\left(\bigvee_{j=k+1}^{k(l+1)} S(t_2)=j\right)$, i.e. for $i>j$

$$\Pr\{S(t_1)=i\wedge S(t_2)=j\wedge P(t_1)=P(t_2)\}=0\ ,\ t_2>t_1 \tag{4.107}$$

because if the states $S(t_1) = i$ and $S(t_2) = j$ belong to the same pulse, i cannot be greater than j.

Further for $i > j$ and $t_1 < t_2$

$$\Pr\{S(t_1) = i \wedge S(t_2) = j \wedge P(t_1) \neq P(t_2)\} = \Pr\{S(t_1) = i \wedge S(t_2) = j\} \tag{4.108}$$

which follows from

$$\Pr\{A \cap B\} = \Pr\{A\} \quad \text{if} \quad A \subset B \tag{4.109}$$

and here $A = \left(\cup_{i=j+1}^{k(l+1)} S(t_1) = i\right) \cap \left(\cup_{j=k+1}^{k(l+1)} S(t_2) = j\right)$ and $B = \{P(t_1) \neq P(t_2)\}$. In other words the situation $i > j$ can only occur if $S(t_1) = i$ and $S(t_2) = j$ belong to different pulses. The probability at the right-hand side of (4.108) is expressed as

$$\Pr\{S(t_1) = i \wedge S(t_2) = j\} = \Pi_{ij}(t_2 - t_1)\mathcal{P}_i(t_1) \tag{4.110}$$

where

$$\Pi_{ij}(t_2 - t_1) = \Pr\left\{S(t_2) = j \middle| S(t_1) = i\right\} \quad , \quad \mathcal{P}_i(t_1) = \Pr\{S(t_1) = i\} \tag{4.111}$$

is just the probability $\mathcal{P}_j(t_2 - t_1)$, evaluated with the following initial condition at $t_2 - t_1 = 0$

$$\mathcal{P}_k(0) = \delta_{ik} = \begin{cases} 1 & , \quad k = i \\ 0 & , \quad k \neq i \end{cases} \tag{4.112}$$

where δ_{ik} is the Kronecker delta.

Finally, for $\left(\bigvee_{i=k+1}^{j} S(t_1) = i\right) \bigwedge \left(\bigvee_{j=k+1}^{k(l+1)} S(t_2) = j\right)$, i.e. $i \leq j$, seeing that this may occur either if $P(t_1) = P(t_2)$ or if $P(t_1) \neq P(t_2)$, which are two exhaustive and mutually exclusive events,

$$\begin{aligned}
&\Pr\{S(t_1) = i \wedge S(t_2) = j\} = \\
&\Pr\{S(t_1) = i \wedge S(t_2) = j \wedge P(t_1) = P(t_2)\} + \\
&\Pr\{S(t_1) = i \wedge S(t_2) = j \wedge P(t_1) \neq P(t_2)\} = \\
&\Pr\{S(t_1) = i \wedge S(t_2) = j\} \Pr\{P(t_1) = P(t_2)\} + \\
&\Pr\{S(t_1) = i \wedge S(t_2) = j\} \Pr\{P(t_1) \neq P(t_2)\} = \\
&\Pr\{S(t_1) = i \wedge S(t_2) = j\} \Pr\{T_d > t_2 - t_1\} + \\
&\Pr\{S(t_1) = i \wedge S(t_2) = j\} \Pr\{T_d < t_2 - t_1\} = \\
&\Pi_{ij}(t_2 - t_1)\mathcal{P}_i(t_1)\Big(1 - F_{T_d}(t_2 - t_1)\Big) + \Pi_{ij}(t_2 - t_1)\mathcal{P}_i(t_1)F_{T_d}(t_2 - t_1)
\end{aligned} \tag{4.113}$$

After using the results (4.107), (4.108) and (4.113) in (4.106) we arrive at the following expression for the auto-correlation function, for $t_1 < t_2$

$$\mu_{XX}(t_1,t_2) = E\left[P^2\right] \sum_{i=k+1}^{j} \sum_{j=k+1}^{k(l+1)} \Pi_{ij}(t_2-t_1)\mathcal{P}_i(t_1)\Big(1-F_{T_d}(t_2-t_1)\Big)+$$

$$E^2[P] \sum_{i=k+1}^{j} \sum_{j=k+1}^{k(l+1)} \Pi_{ij}(t_2-t_1)\mathcal{P}_i(t_1)F_{T_d}(t_2-t_1)+E^2[P] \sum_{i=j+1}^{k(l+1)} \sum_{j=k+1}^{k(l+1)} \Pi_{ij}(t_2-t_1)\mathcal{P}_i(t_1) \tag{4.114}$$

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4.5 Example problems

4.1. Evaluate the ordinary renewal density for an Erlang process with $k = 2$, $k = 3$ and $k = 4$.

4.2. Evaluate the cumulants of the filtered Poisson process, by differentiating the log-characteristic function.

4.3. For a train of non-overlapping rectangular pulses with $k = 1$ and $l = 1$, $k = 1$ and $l = 2$, $k = 2$ and $l = 1$, do the following:

- evaluate the probability density and distribution functions of a truncated pulse duration and of a time gap between the consecutive pulses,
- draw the chain of Markov states,
- formulate the set of differential equations governing the Markov states probabilities,
- evaluate "on" and "off" probabilities, assuming different initial conditions.

CHAPTER 5

DYNAMIC RESPONSE OF NON-LINEAR SYSTEMS TO POISSON IMPULSE PROCESS EXCITATIONS. NON-DIFFUSIVE MARKOV PROCESS TECHNIQUES

5.1 Governing stochastic integro-differential equations

Consider a multi-degree-of-freedom non-linear dynamical system subjected to a train of impulses driven by a Poisson process. Let us confine attention to the univariate Poisson process. The system state variables are governed, in a general case of the excitation, by the equation

$$\frac{d}{dt}\mathbf{Z}(t) = \mathbf{c}\big(\mathbf{Z}(t),t\big) + \sum_{i=1}^{N(t)} \mathbf{b}\big(\mathbf{Z}(t),t,P_i\big)\delta(t-t_i) =$$
$$\mathbf{c}\big(\mathbf{Z}(t),t\big) + \sum_{i=1}^{\infty} \mathbf{b}\big(\mathbf{Z}(t),t,P_i\big)\delta(t-t_i)\hat{\mathbf{1}}(t-t_i) \tag{5.1}$$

where $\hat{\mathbf{1}}(t-t_i)$ is the indicator function defined by (1.16). In general, the initial conditions can be non-zero and random, given by the vector random variable

$$\mathbf{Z}(0) = \mathbf{Z}_0 \tag{5.2}$$

Assumptions about the train of impulses, i.e. about the the counting process $\{N(t), t \in [0,\infty[\}$ and about the random variables P_i are the same as in the section 1.1.2.

It is assumed that the compound Poisson process characterized by the counting process $\{N(t), t \in [0,\infty[\}$ and by the random variables P_i is statistically independent of the initial conditions, i.e. of the random vector $\mathbf{Z}_0$.

Integration of the governing equation (5.1) over the time, with regard to the Dirac delta function in the integrand, leads to the following equation

$$\mathbf{Z}(t) = \mathbf{Z}_0 + \int_0^t \mathbf{c}\big(\mathbf{Z}(\tau),\tau\big)d\tau + \sum_{i=1}^{\infty} \mathbf{b}\big(\mathbf{Z}(t_i),t_i,P_i\big)\hat{\mathbf{1}}(t-t_i)$$
$$= \mathbf{Z}_0 + \int_0^t \mathbf{c}\big(\mathbf{Z}(\tau),\tau\big)d\tau + \sum_{i=1}^{N(t)} \mathbf{b}\big(\mathbf{Z}(t_i),t_i,P_i\big) \tag{5.3}$$

where the last term (the excitation term), is just analogous to the filtered Poisson process and can be substituted by an integral with respect to the Poisson random measure, cf. (1.15), (1.54). This results in the following integral equation

$$\mathbf{Z}(t) = \mathbf{Z}_0 + \int_0^t \mathbf{c}\big(\mathbf{Z}(\tau),\tau\big)d\tau + \int_0^t \int_{\mathcal{P}} \mathbf{b}\big(\mathbf{Z}(\tau),\tau,p\big)M(d\tau,\tau,dp,p) \tag{5.4}$$

whose equivalent integro-differential form, slightly more general than (2.78), is

$$d\mathbf{Z}(t) = \mathbf{c}\big(\mathbf{Z}(t),t\big)dt + \int_{\mathcal{P}} \mathbf{b}\big(\mathbf{Z}(t),t,p\big)M(dt,t,dp,p) \tag{5.5}$$

and the corresponding differential form is

$$d\mathbf{Z}(t) = \mathbf{c}\big(\mathbf{Z}(t),t\big)dt + \mathbf{b}\big(\mathbf{Z}(t),t,P(t)\big)dN(t) \tag{5.6}$$

together with the initial conditions (5.2).

Hence the state vector $\mathbf{Z}(t)$ of the dynamical system subjected to a Poisson-distributed train of impulses is, provided that the impulse process is statistically independent of the initial conditions, a *non-diffusive Markov process*, the so-called *Poisson-driven Markov process.*

In what follows let us assume, for the sake of simplicity that the process has zero initial conditions.

5.2 Available solution for the probability density of the response

The possibilities of solving the equation for the response probability density of the Poisson driven systems are even more limited as it is in the case of Gaussian white noise driven systems. Renger [5.2] presented an exact solution in the case of a linear first order filter (single state variable system), but no exact solutions are known for vibratory systems, not even in the case of a simple linear oscillator. Roberts [5.3] devised a perturbation technique, the use of which was only shown for a linear oscillator. The perturbation technique due to Cai and Lin [5.6] which is outlined herein is more general and is applicable to the non-linear problems belonging to the class of generalized stationary potential, i.e. the class for which the exact stationary solution of the usual Fokker-Planck equation can be obtained, cf. Lin and Cai [5.4] and Cai and Lin [5.5].

Consider a SDOF non-linear system governed by the stochastic equations

$$\left.\begin{aligned} dZ_1 &= Z_2 dt \\ dZ_2 &= c_2(Z_1,Z_2)dt + b_2(Z_1,Z_2)P(t)dN(t) \end{aligned}\right\} \tag{5.7}$$

where, of course, $c_1(Z_1,Z_2) = Z_2$, $b_1 = 0$.

The forward Kolmogorov-Feller equation governing the response probability density $f_{\{\mathbf{Z}\}}(z_1,z_2,t)$ is represented as the following Kramer-Moyal expansion of the integro-differential operator, cf. (2.87)

$$\frac{\partial}{\partial t}f_{\{\mathbf{Z}\}}(z_1,z_2,t)=$$
$$-z_2\frac{\partial f_{\{\mathbf{Z}\}}(z_1,z_2,t)}{\partial z_1}-\frac{\partial}{\partial z_2}\Big(c_2(z_1,z_2)f_{\{\mathbf{Z}\}}(z_1,z_2,t)\Big)+\frac{1}{2!}\nu E[P^2]\frac{\partial^2}{\partial z_2^2}\Big(b_2^2(z_1,z_2)f_{\{\mathbf{Z}\}}(z_1,z_2,t)\Big)$$
$$-\frac{1}{3!}\nu E[P^3]\frac{\partial^3}{\partial z_2^3}\Big(b_2^3(z_1,z_2)f_{\{\mathbf{Z}\}}(z_1,z_2,t)\Big)+\frac{1}{4!}\nu E[P^4]\frac{\partial^4}{\partial z_2^4}\Big(b_2^4(z_1,z_2)f_{\{\mathbf{Z}\}}(z_1,z_2,t)\Big)-\cdots \tag{5.8}$$

In order to devise the perturbation scheme let us introduce the perturbation parameter through the relation

$$\varepsilon^n I_n=\nu E[P^{n+2}],\quad n=0,1,2,\ldots \tag{5.9}$$

where I_n are finite constants and $\varepsilon\to 0$ as $\nu\to\infty$. At the same time $E[P^2]$ must tend to zero in such a way that $\nu E[P^2]$ is kept constant and the impulse process approaches the Gaussian white noise. Let us choose the perturbation parameter as $\varepsilon=\nu^{-1/2}$, i.e. $\varepsilon^{-2}=\nu$. Then $E[P^2]$, $E[P^3]$ and $E[P^4]$ are of orders ε^2, ε^3 and ε^4, respectively.

The solution $f_{\{\mathbf{Z}\}}(z_1,z_2)$ of the stationary counterpart of the equation (5.8) is tried in the form of the regular perturbation expansion in the perturbation parameter ε

$$f_{\{\mathbf{Z}\}}(z_1,z_2)=f^{(0)}_{\{\mathbf{Z}\}}(z_1,z_2)+\varepsilon f^{(1)}_{\{\mathbf{Z}\}}(z_1,z_2)+\varepsilon^2 f^{(2)}_{\{\mathbf{Z}\}}(z_1,z_2)+\cdots \tag{5.10}$$

Upon inserting this expansion into the governing equation (stationary counterpart of (5.8)) and collecting the terms of the same power of ε the following set of second order partial differential equations are obtained

$$-z_2\frac{\partial f^{(0)}_{\{\mathbf{Z}\}}(z_1,z_2)}{\partial z_1}-\frac{\partial}{\partial z_2}\Big(c_2(z_1,z_2)f^{(0)}_{\{\mathbf{Z}\}}(z_1,z_2)\Big)+\frac{I_0}{2!}\frac{\partial^2}{\partial z_2^2}\Big(b_2^2(z_1,z_2)f^{(0)}_{\{\mathbf{Z}\}}(z_1,z_2)\Big)=0 \tag{5.11}$$

$$-z_2\frac{\partial f^{(1)}_{\{\mathbf{Z}\}}(z_1,z_2)}{\partial z_1}-\frac{\partial}{\partial z_2}\Big(c_2(z_1,z_2)f^{(1)}_{\{\mathbf{Z}\}}(z_1,z_2)\Big)+\frac{I_0}{2!}\frac{\partial^2}{\partial z_2^2}\Big(b_2^2(z_1,z_2)f^{(1)}_{\{\mathbf{Z}\}}(z_1,z_2)\Big)=$$
$$\frac{I_1}{3!}\frac{\partial^3}{\partial z_2^3}\Big(b_2^3(z_1,z_2)f^{(0)}_{\{\mathbf{Z}\}}(z_1,z_2)\Big) \tag{5.12}$$

$$-z_2\frac{\partial f^{(2)}_{\{\mathbf{Z}\}}(z_1,z_2)}{\partial z_1}-\frac{\partial}{\partial z_2}\Big(c_2(z_1,z_2)f^{(2)}_{\{\mathbf{Z}\}}(z_1,z_2)\Big)+\frac{I_0}{2!}\frac{\partial^2}{\partial z_2^2}\Big(b_2^2(z_1,z_2f^{(2)}_{\{\mathbf{Z}\}}(z_1,z_2)\Big)=$$
$$\frac{I_1}{3!}\frac{\partial^3}{\partial z_2^3}\Big(b_2^3(z_1,z_2)f^{(1)}_{\{\mathbf{Z}\}}(z_1,z_2)\Big)-\frac{I_2}{4!}\frac{\partial^4}{\partial z_2^4}\Big(b_2^4(z_1,z_2)f^{(0)}_{\{\mathbf{Z}\}}(z_1,z_2)\Big) \tag{5.13}$$

The equation (5.11) is the usual stationary Fokker-Planck equation, which can be solved exactly if the problem belongs to the class of generalized stationary potential. The solution has the general form, cf. section 3.1.2.

$$f^{(0)}_{\{\mathbf{Z}\}}(z_1, z_2) = C_0 \exp(-\Phi) \tag{5.14}$$

where C_0 is the normalization constant and Φ is the suitable potential function. It is expedient to represent the second term of the expansion (5.10) as

$$f^{(1)}_{\{\mathbf{Z}\}}(z_1, z_2) = f^{(0)}_{\{\mathbf{Z}\}}(z_1, z_2) Q^{(1)}_{\{\mathbf{Z}\}}(z_1, z_2) \tag{5.15}$$

Upon substituting (5.15) into equation (5.12) and subtracting the appearing Fokker-Planck-Kolmogorov equation for $f^{(0)}_{\{\mathbf{Z}\}}(z_1, z_2)$, the following equation for $Q^{(1)}_{\{\mathbf{Z}\}}(z_1, z_2)$ is arrived at

$$\begin{aligned} &-z_2 \frac{\partial Q^{(1)}_{\{\mathbf{Z}\}}(z_1, z_2)}{\partial z_1} + \left(2I_0 b_2(z_1, z_2) \frac{\partial b_2(z_1, z_2)}{\partial z_2} - I_0 b_2^2(z_1, z_2) \frac{\partial \Phi}{\partial z_2} + c_2(z_1, z_2) \right) \frac{\partial Q^{(1)}_{\{\mathbf{Z}\}}(z_1, z_2)}{\partial z_2} \\ &+ \frac{I_0}{2!} b_2^2(z_1, z_2) \frac{\partial^2 Q^{(1)}_{\{\mathbf{Z}\}}(z_1, z_2)}{\partial z_2^2} = \frac{I_1}{3! f^{(0)}_{\{\mathbf{Z}\}}(z_1, z_2)} \frac{\partial^3}{\partial z_2^3} \left(b_2^3(z_1, z_2) f^{(0)}_{\{\mathbf{Z}\}}(z_1, z_2) \right) \end{aligned} \tag{5.16}$$

Although this equation is difficult to solve for arbitrary $c_2(z_1, z_2)$ and $b_2(z_1, z_2)$, in the case of polynomial form of the potential Φ and constant or linear $b_2(z_1, z_2)$ the solution of the following equation

$$\begin{aligned} &- z_2 \frac{\partial Q^{(1)}_{\{\mathbf{Z}\}}(z_1, z_2)}{\partial z_1} + \left(A z_1 + B z_2 \right) \frac{\partial Q^{(1)}_{\{\mathbf{Z}\}}(z_1, z_2)}{\partial z_2} + \\ &\frac{I_0}{2!} b_2^2(z_1, z_2) \frac{\partial^2 Q^{(1)}_{\{\mathbf{Z}\}}(z_1, z_2)}{\partial z_2^2} = \frac{I_1}{3! f^{(0)}_{\{\mathbf{Z}\}}(z_1, z_2)} \frac{\partial^3}{\partial z_2^3} \left(b_2^3(z_1, z_2) f^{(0)}_{\{\mathbf{Z}\}}(z_1, z_2) \right) \end{aligned} \tag{5.17}$$

is available and has polynomial form. The latter equation may be considered as equivalent to the former, if the constants A and B are suitably chosen. The procedure is similar to that of the equivalent linearization scheme. It is required that the error, i.e. the difference of equations (5.16) and (5.17)

$$\epsilon = A Z_1 + B Z_2 - \left(2I_0 b_2(Z_1, Z_2) \frac{\partial b_2(Z_1, Z_2)}{\partial Z_2} - I_0 b_2^2(Z_1, Z_2) \frac{\partial \Phi}{\partial Z_2} + c_2(Z_1, Z_2) \right) \tag{5.18}$$

be minimized in the mean square sense, i.e.

$$\frac{\partial}{\partial A} E\left[\epsilon^2\right] = 0 \quad , \quad \frac{\partial}{\partial B} E\left[\epsilon^2\right] = 0 \tag{5.19}$$

which leads to the following solutions

$$A = \frac{E\left[Z_1\left(2I_0 b_2(Z_1,Z_2)\frac{\partial b_2(Z_1,Z_2)}{\partial Z_2} - I_0 b_2^2(Z_1,Z_2)\frac{\partial \Phi}{\partial Z_2} + c_2(Z_1,Z_2)\right)\right]}{E\left[Z_1^2\right]} \tag{5.20}$$

$$B = \frac{E\left[Z_2\left(2I_0 b_2(Z_1,Z_2)\frac{\partial b_2(Z_1,Z_2)}{\partial Z_2} - I_0 b_2^2(Z_1,Z_2)\frac{\partial \Phi}{\partial Z_2} + c_2(Z_1,Z_2)\right)\right]}{E\left[Z_2^2\right]} \tag{5.21}$$

The stationary, first-order perturbation solution then becomes

$$f_{\{\mathbf{Z}\}}(z_1,z_2) = f^{(0)}_{\{\mathbf{Z}\}}(z_1,z_2)\left(1+\varepsilon Q^{(1)}_{\{\mathbf{Z}\}}(z_1,z_2)\right) \tag{5.22}$$

The second order term may be obtained by assuming

$$f^{(2)}_{\{\mathbf{Z}\}}(z_1,z_2) = f^{(0)}_{\{\mathbf{Z}\}}(z_1,z_2)Q^{(2)}_{\{\mathbf{Z}\}}(z_1,z_2) \tag{5.22}$$

and by proceeding similarly. The equation for $Q^{(2)}_{\{\mathbf{Z}\}}(z_1,z_2)$ is obtained as

$$-z_2\frac{\partial Q^{(2)}_{\{\mathbf{Z}\}}(z_1,z_2)}{\partial z_1} + \left(2I_0 b_2(z_1,z_2)\frac{\partial b_2(z_1,z_2)}{\partial z_2} - I_0 b_2^2(z_1,z_2)\frac{\partial \Phi}{\partial z_2} + c_2(z_1,z_2)\right)\frac{\partial Q^{(2)}_{\{\mathbf{Z}\}}(z_1,z_2)}{\partial z_2} +$$

$$\frac{I_0}{2!}b_2^2(z_1,z_2)\frac{\partial^2 Q^{(2)}_{\{\mathbf{Z}\}}(z_1,z_2)}{\partial z_2^2} = \frac{I_1}{3!f^{(0)}_{\{\mathbf{Z}\}}(z_1,z_2)}\frac{\partial^3}{\partial z_2^3}\left(b_2^3(z_1,z_2)f^{(0)}_{\{\mathbf{Z}\}}(z_1,z_2)Q^{(1)}_{\{\mathbf{Z}\}}(z_1,z_2)\right) -$$

$$\frac{I_2}{4!f^{(0)}_{\{\mathbf{Z}\}}(z_1,z_2)}\frac{\partial^4}{\partial z_2^4}\left(b_2^4(z_1,z_2)f^{(0)}_{\{\mathbf{Z}\}}(z_1,z_2)\right) \tag{5.24}$$

The function $Q^{(2)}_{\{\mathbf{Z}\}}(z_1,z_2)$ is evaluated in the same way as $Q^{(1)}_{\{\mathbf{Z}\}}(z_1,z_2)$. The second-order perturbation solution is

$$f_{\{\mathbf{Z}\}}(z_1,z_2) = f^{(0)}_{\{\mathbf{Z}\}}(z_1,z_2)\left(1+\varepsilon Q^{(1)}_{\{\mathbf{Z}\}}(z_1,z_2)+\varepsilon^2 Q^{(2)}_{\{\mathbf{Z}\}}(z_1,z_2)\right) \tag{5.25}$$

The procedure as described above can be continued to yield the higher-order corrections. Moreover the results can be improved by repeated, iterative use of the procedure. The obtained approximate solution $f_{\{\mathbf{Z}\}}(z_1,z_2)$ may be assumed as initial estimate in the next iteration instead of $f^{(0)}_{\{\mathbf{Z}\}}(z_1,z_2)$. However, convergence can only be expected if $\epsilon = \nu^{1/2}$ is small, i.e. in the case of relatively dense pulse trains.

5.3 Differential equations governing the response moments and truncation technique

5.3.1 Differential equations for moments and for cumulants

Averaging of the stochastic equations, with taking into account the properties (1.50) of the Poisson random measure leads to the equations for the mean values $E[Z_i(t)] = \mu_i(t)$ in the form of

$$\dot{\mu}_i(t) = E\big[c_i(\mathbf{Z}(t),t)\big] + \nu(t)\int_{\mathcal{P}} E\Big[b_i\Big(\mathbf{Z}(t),t,p\Big)\Big]\, f_P(p)dp =$$

$$E\big[c_i(\mathbf{Z}(t),t)\big] + \nu(t)E\big[b_i\big(\mathbf{Z}(t),t,P(t)\big)\big] \tag{5.26}$$

where $P(t)$ is the random magnitude of the impulse arriving in $[t, t+dt[$.

Equations for the centralized state variables (zero-mean response processes) $Z_i^0(t) = Z_i(t) - \mu_i(t)$ are obtained as

$$d\mathbf{Z}^0(t) = \mathbf{c}^0\left(\mathbf{Z}^0(t),t\right)dt + \int_{\mathcal{P}} \mathbf{b}(\mathbf{Z}(t),t,p)M(dt,t,dp,p) \tag{5.27}$$

where

$$\mathbf{c}^0\big(\mathbf{Z}^0(t),t\big) = \mathbf{c}\left(\mathbf{Z}^0(t)+\boldsymbol{\mu}(t),t\right) - E\left[\mathbf{c}\left(\mathbf{Z}^0(t)+\boldsymbol{\mu}(t),t\right)\right] - \nu(t)E\Big[\mathbf{b}\Big(\mathbf{Z}(t),t,P\Big)\Big] \tag{5.28}$$

Let us denote the nth order joint central moments of the state variables as

$$\kappa_{i_1\cdots i_n}(t) = E\left[\prod_{q=1}^{n} Z_{i_q}^0(t)\right] \tag{5.29}$$

Equations for moments are obtained from the differential rule (2.88) for the function $f(\mathbf{Z}^0(t)) = \prod_{q=1}^{n} Z_{i_q}^0(t)$. After performing the expectation the general equation for the joint central moments is obtained as, cf (2.65)

$$\frac{d}{dt}\kappa_{i_1\ldots i_n}(t) = E\Big[\frac{\partial}{\partial t}\Big(Z_{i_1}^0(t)\cdots Z_{i_n}^0(t)\Big)\Big] + E\Big[\mathcal{K}_{\mathbf{z},t}^T\big[Z_{i_1}^0(t)\cdots Z_{i_n}^0(t)\big]\Big] =$$

$$\sum_i E\left[c_i^0\left(\mathbf{Z}^0(t),t\right)\frac{\partial}{\partial Z_i^0}\left(Z_{i_1}^0(t)\cdots Z_{i_n}^0(t)\right)\right] +$$

$$\nu(t)\int_{\mathcal{P}} E\Big[\Big(Z_{i_1}^0(t)+b_{i_1}\left(\mathbf{Z}(t),p\right)\Big)\cdots\Big(Z_{i_n}^0(t)+b_{i_n}\left(\mathbf{Z}(t),p\right)\Big) -$$

$$\Big(Z_{i_1}^0(t)\cdots Z_{i_n}^0(t)\Big)\Big] f_P(p)dp \tag{5.30}$$

In particular the equations for the joint second-, third- and fourth-order moments, are obtained in the form of

$$\dot{\kappa}_{ij}(t) = 2\Big\{E\Big[Z_i^0\big(c_j^0(\mathbf{Z}^0(t)) + \nu(t)b_j(\mathbf{Z}(t),P)\big)\Big]\Big\}_s + \nu(t)E\Big[b_i(\mathbf{Z}(t),P)b_j(\mathbf{Z}(t),P)\Big] \tag{5.31}$$

$$\dot{\kappa}_{ijk}(t) = 3\Big\{E\Big[Z_i^0 Z_j^0\Big(c_k^0(\mathbf{Z}^0(t)) + \nu(t)b_k(\mathbf{Z}(t),P)\Big)\Big]\Big\}_s + 3\nu(t)\Big\{E\Big[Z_i^0 b_j(\mathbf{Z}(t),P)b_k(\mathbf{Z}(t),P)\Big]\Big\}_s + \nu(t)E\Big[b_i(\mathbf{Z}(t),P)b_j(\mathbf{Z}(t),P)b_k(\mathbf{Z}(t),P)\Big] \tag{5.32}$$

$$\dot{\kappa}_{ijkl}(t) = 4\Big\{E\Big[Z_i^0 Z_j^0 Z_k^0\Big(c_l^0(\mathbf{Z}^0(t)) + \nu(t)b_l(\mathbf{Z}(t),P)\Big)\Big]\Big\}_s + 6\nu(t)\Big\{E\Big[Z_i^0 Z_j^0 b_k(\mathbf{Z}(t),P)b_l(\mathbf{Z}(t),P)\Big]\Big\}_s + 4\nu(t)\Big\{E\Big[Z_i^0 b_j(\mathbf{Z}(t),P)b_k(\mathbf{Z}(t),P)b_l(\mathbf{Z}(t),P)\Big]\Big\}_s + \nu(t)E\Big[b_i(\mathbf{Z}(t),P)b_j(\mathbf{Z}(t),P)b_k(\mathbf{Z}(t),P)b_l(\mathbf{Z}(t),P)\Big] \tag{5.33}$$

If the vector $\mathbf{b}(\mathbf{Z}(t),t,P) \equiv \mathbf{b}P$, i.e. it is state independent and constant, the equations for moments (5.31)–(5.33) simplify to

$$\dot{\kappa}_{ij}(t) = 2\Big\{E\Big[Z_i^0\Big(c_j^0(\mathbf{Z}^0(t)) + \nu(t)b_jP\Big)\Big]\Big\}_s + b_ib_j\nu(t)E\left[P^2\right] \tag{5.34}$$

$$\dot{\kappa}_{ijk}(t) = 3\Big\{E\Big[Z_i^0 Z_j^0\Big(c_k^0(\mathbf{Z}^0(t)) + \nu(t)b_kP\Big)\Big]\Big\}_s + b_ib_jb_k\nu(t)E\left[P^3\right] \tag{5.35}$$

$$\dot{\kappa}_{ijkl}(t) = 4\Big\{E\Big[Z_i^0 Z_j^0 Z_k^0\Big(c_l^0(\mathbf{Z}^0(t)) + \nu(t)b_lP\Big)\Big]\Big\}_s + 6\left\{b_ib_j\kappa_{kl}(t)\right\}_s\nu(t)E\left[P^2\right] + b_ib_jb_kb_l\nu(t)E\left[P^4\right] \tag{5.36}$$

The differential equations for moments must be associated by initial conditions pertinent to (5.2).

Alternatively the equations for joint central moments may be derived from the differential equation governing the time evolution of the joint characteristic function $\Phi_{\{\mathbf{Z}^0\}}(\boldsymbol{\theta},t) = E\left[\exp\left(i\theta_kZ_k^0(t)\right)\right]$ of the centralized state variables. Here and in what follows the summation convention is applied over dummy indices. The latter equation may be obtained

from the differential rule (2.88) for the function $f(\mathbf{Z}^0(t)) = \exp\left(i\theta_k Z_k^0(t)\right)$. Next, the equations for moments are obtained by differentiation as

$$\frac{d}{dt}\kappa_{i_1\ldots i_n}(t) = \frac{d}{dt}\, i^{-n}\frac{\partial^n \Phi_{\{\mathbf{Z}^0\}}(\boldsymbol{\theta},t)}{\partial\theta_{i_1}\cdots\partial\theta_{i_n}}\bigg|_{\theta_{i_1}=\cdots=\theta_{i_n}=0} = i^{-n}\,\frac{\partial^n}{\partial\theta_{i_1}\cdots\partial\theta_{i_n}}\frac{d\Phi_{\{\mathbf{Z}^0\}}(\boldsymbol{\theta},t)}{dt}\bigg|_{\theta_{i_1}=\cdots=\theta_{i_n}=0} \tag{5.37}$$

In order to obtain the equations governing the joint cumulants, the equation for the time evolution of the log-characteristic function $\Lambda_{\{\mathbf{Z}^0\}}(\boldsymbol{\theta},t) = \ln\Phi_{\{\mathbf{Z}^0\}}(\boldsymbol{\theta},t)$ is first derived. This may be done as follows. The increment $d\Lambda_{\{\mathbf{Z}^0\}}(\boldsymbol{\theta},t) = \Lambda_{\{\mathbf{Z}^0\}}(\boldsymbol{\theta},t+dt) - \Lambda_{\{\mathbf{Z}^0\}}(\boldsymbol{\theta},t)$ of the log-characteristic function during the time interval $[t, t+dt[$ equals

$$d\Lambda_{\{\mathbf{Z}^0\}}(\boldsymbol{\theta},t) = \ln E\Big[\exp\Big(i\theta_k\big(Z_k^0(t)+dZ_k^0(t)\big)\Big)\Big] - \ln E\left[\exp\left(i\theta_k Z_k^0(t)\right)\right] =$$

$$\ln E\left[\exp\left(i\theta_k Z_k^0(t)\right)\exp\left(i\theta_k dZ_k^0(t)\right)\right] - \ln E\left[\exp\left(i\theta_k Z_k^0(t)\right)\right] \tag{5.38}$$

Using equation (5.27) one obtains

$$\exp\left(i\theta_k dZ_k^0(t)\right) = \exp\left(i\theta_k\left(c_k^0\left(\mathbf{Z}^0(t),t\right)dt + \int_{\mathcal{P}} b_k\big(\mathbf{Z}(t),t,p\big)M(dt,t,dp,p)\right)\right) =$$

$$\exp\Big(i\theta_k c_k^0\big(\mathbf{Z}^0(t),t\big)dt\Big)\exp\left(i\theta_l\int_{\mathcal{P}} b_l\big(\mathbf{Z}(t),t,p\big)M(dt,t,dp,p)\right) \tag{5.39}$$

Expanding the exponentials in Taylor series and neglecting the terms of orders higher than dt yields

$$\exp\left(i\theta_k dZ_k^0(t)\right) =$$

$$\left(1+i\theta_k c_k^0\left(\mathbf{Z}^0(t),t\right)dt\right)\left(1+\sum_{n=1}^{\infty}\frac{1}{n!}\left(i\theta_l\int_{\mathcal{P}} b_l\big(\mathbf{Z}^0(t),t,p\big)M(dt,t,dp,p)\right)^n\right) =$$

$$1+i\theta_k c_k^0\left(\mathbf{Z}^0(t),t\right)dt+$$

$$\sum_{n=1}^{\infty}\left(\frac{i^n}{n!}\theta_{i_1}\cdots\theta_{i_n}\int_{\mathcal{P}} b_{i_1}\big(\mathbf{Z}(t),t,p\big)\cdots b_{i_n}\big(\mathbf{Z}(t),t,p\big)M(dt,t,dp,p)\right) \tag{5.40}$$

where the property (1.53) of the measure $M(dt,t,dp,p)$ is applied in the last statement.

Consequently we have

$$d\Lambda_{\{\mathbf{Z}^0\}}(\boldsymbol{\theta},t) = \ln E\Big[\exp\Big(i\theta_k Z_k^0(t)\Big)\Big(1 + i\theta_l c_l^0\big(\mathbf{Z}^0(t),t\big)dt\, +$$

$$\sum_{n=1}^{\infty}\Big(\frac{i^n}{n!}\theta_{i_1}\cdots\theta_{i_n}\int_{\mathcal{P}} b_{i_1}\big(\mathbf{Z}(t),t,p\big)\cdots b_{i_n}\big(\mathbf{Z}(t),t,p\big)M(dt,t,dp,p)\Big)\Big)\Big] -$$

$$\ln E\Big[\exp\big(i\theta_k Z_k^0(t)\big)\Big] =$$

$$\ln\Bigg\{1 + \frac{1}{\Phi_{\{\mathbf{Z}^0\}}(\boldsymbol{\theta},t)}E\Big[\exp\Big(i\theta_k Z_k^0(t)\Big)\Big(i\theta_l c_l^0\big(\mathbf{Z}^0(t),t\big)dt +$$

$$\sum_{n=1}^{\infty}\Big(\frac{i^n}{n!}\theta_{i_1}\cdots\theta_{i_n}\int_{\mathcal{P}} b_{i_1}\big(\mathbf{Z}(t),t,p\big)\cdots b_{i_n}\big(\mathbf{Z}(t),t,p\big)M(dt,t,dp,p)\big)\Big)\Big]\Bigg\} =$$

$$\ln\Bigg\{1 + \frac{1}{\Phi_{\{\mathbf{Z}^0\}}(\boldsymbol{\theta},t)}\Big(E\Big[\exp\big(i\theta_k Z_k^0(t)\big)i\theta_l c_l^0\big(\mathbf{Z}^0(t),t\big)\Big] +$$

$$\sum_{n=1}^{\infty}\frac{i^n}{n!}\theta_{i_1}\cdots\theta_{i_n}E\Big[\exp\big(i\theta_k Z_k^0(t)\big)\,b_{i_1}\big(\mathbf{Z}(t),t,P\big)\cdots b_{i_n}\big(\mathbf{Z}(t),t,P\big)\Big]\nu(t)\Big)dt\Bigg\} \quad (5.41)$$

Using again Taylor expansion and neglecting the terms of orders higher than dt gives

$$\Phi_{\{\mathbf{Z}^0\}}(\boldsymbol{\theta},t)\frac{d\Lambda_{\{\mathbf{Z}^0\}}(\boldsymbol{\theta},t)}{dt} = E\Big[\exp\Big(i\theta_k Z_k^0(t)\Big)i\theta_l c_l^0\big(\mathbf{Z}^0(t),t\big)\Big] +$$

$$\sum_{n=1}^{\infty}\frac{i^n}{n!}\theta_{i_1}\cdots\theta_{i_n}E\Big[\exp\Big(i\theta_k Z_k^0(t)\Big)b_{i_1}\big(\mathbf{Z}(t),t,P\big)\cdots b_{i_n}\big(\mathbf{Z}(t),t,P\big)\Big]\nu(t) \quad (5.42)$$

Equations for the joint cumulants $\lambda_{i_1\ldots i_n}(t)$ are obtained from the above equation with the help of the relationship

$$\frac{d}{dt}\lambda_{i_1\ldots i_n}(t) = \frac{d}{dt}\; i^{-n}\frac{\partial^n\Lambda_{\{\mathbf{Z}^0\}}(\boldsymbol{\theta},t)}{\partial\theta_{i_1}\cdots\partial\theta_{i_n}}\Bigg|_{\theta_{i_1}=\cdots=\theta_{i_n}=0} =$$

$$i^{-n}\;\frac{\partial^n}{\partial\theta_{i_1}\cdots\partial\theta_{i_n}}\;\frac{d\Lambda_{\{\mathbf{Z}^0\}}(\boldsymbol{\theta},t)}{dt}\Bigg|_{\theta_{i_1}=\cdots=\theta_{i_n}=0} \quad (5.43)$$

The differential equations for the cumulants can also be obtained by differentiation of the identities (3.56) between the central moments and the cumulants.

5.3.2 Truncation of the hierarchy of moment equations. Modified cumulant-neglect closure technique for Poisson impulses problems.

Equations for moments involve unknown expectations of non-linear functions of the state variables. Forms of these functions depend on the forms of non-linear functions $c_i(\mathbf{Z}(t),t)$ and $b_i(\mathbf{Z}(t),t,P(t))$ in the governing equations of motion (5.1). Equations for moments only form a closed set if $c_i(\mathbf{Z}(t),t)$ and $b_i(\mathbf{Z}(t),t,P(t))$ are linear forms in the state variables $\mathbf{Z}(t)$. When this is a polynomial of degree $r>1$, then the equations for moments form an infinite hierarchy, i.e. the equations for the moments of up to nth order involve the moments up to and including the $(n+r-1)$th order. If $c_i(\mathbf{Z}(t),t)$ is of other form than a polynomial in the state variables, then the pertinent expectations appearing in the equations for moments cannot be expressed explicitly in terms of moments. They must be evaluated by performing the integrals with respect to the probability density function. The exact density function is, however, unknown and the approximate, tentative, density function must be assumed. A non-Gaussian tentative density function is often assumed in form of a truncated Gram-Charlier expansion. As indicated in section 3.2.2 this approach is equivalent to using a quasi-moment neglect closure scheme in case of polynomial non-linearities. The accuracy of the results obtained, i.e. of approximate response moments depends on how close to the exact one the assumed tentative density function is. Therefore some modifications of the tentative density function have been proposed in various problems.

Let us consider the case of a polynomial non-linearity and let us study in detail the case of the 3rd order polynomial. Moreover let us concentrate attention on the case of external excitation, on the case of the state independent and constant in time vector $\mathbf{b}(\mathbf{Z}(t),t))=\mathbf{b}$ (constant). The drift term is, cf. (5.28)

$$c_i^0(\mathbf{Z}) = A_i + B_{im}Z_m^0 + C_{imn}Z_m^0Z_n^0 + D_{imnp}Z_m^0Z_n^0Z_p^0 - \nu(t)E[P] \tag{5.44}$$

where $A_i = -C_{imn}\kappa_{mn} - D_{imnp}\kappa_{mnp}$.

Equations (5.34)–(5.36) for the joint central moments take, respectively, the form of

$$\dot{\kappa}_{ij}(t) = 2\left\{B_{im}\kappa_{mj}\right\}_s + 2\left\{C_{imn}\kappa_{mnj}\right\}_s + 2\left\{D_{imnp}\kappa_{mnpj}\right\}_s + \nu(t)E[P^2]b_ib_j \tag{5.45}$$

$$\dot{\kappa}_{ijk}(t) = 3\{A_i\kappa_{jk}\}_s + 3\{B_{im}\kappa_{mjk}\}_s + 3\{C_{imn}\kappa_{mnjk}\}_s +$$

$$3\{D_{imnp}\kappa_{mnpjk}\}_s + \nu(t)E[P^3]b_ib_jb_k \tag{5.46}$$

$$\dot{\kappa}_{ijkl}(t) = 4\{A_i\kappa_{jkl}\}_s + 4\{B_{im}\kappa_{mjkl}\}_s + 4\{C_{imn}\kappa_{mnjkl}\}_s +$$

$$4\{D_{imnp}\kappa_{mnpjkl}\}_s + 6\nu(t)E[P^2]\{\kappa_{ij}b_kb_l\}_s + \nu(t)E[P^4]b_ib_jb_kb_l \tag{5.47}$$

If we decide to truncate the moment equations at a given level, all higher-order redundant moments have to be expressed approximately in terms of lower-order moments,

for example with the help of the cumulant neglect closure, see eq. (3.65) in subsection 3.2.3.

Consider the dynamical system subjected to a random train of impulses and to initial conditions $\mathbf{Z}(0) = \mathbf{z}_0$. If in the time interval $[0, t[$ no impulse occurred, the system has been performing the deterministic drift motion from the initial state $\mathbf{z}_0$ at the time 0 to the state $\mathbf{z}(t) = \mathbf{e}(t|\mathbf{z}_0, 0)$ at the time t, or it has been at rest since the initial instant, in the case of zero initial conditions. Notice that $\mathbf{e}(0|\mathbf{z}_0, 0) = \mathbf{z}_0$.

If the train of impulses is driven by a homogeneous Poisson process, the probability P_0 of no impulse occurrence in the time interval $[0, t[$ is expressed as, cf. (1.22)

$$P_0 = \Pr\{N(t) = 0\} = \exp(-\nu t) \tag{5.48}$$

The probability P_0 may be high, close to the unity, if the length t of the time interval is small, i.e. at the early transient stage, especially if also the mean arrival rate ν is small.

Let us introduce two expected values: unconditional one $\boldsymbol{\mu}(t) = E[\mathbf{Z}(t)]$ and the conditional expected value, given that at least one impulse occurred, denoted as $\boldsymbol{\mu}^0(t) = E[\mathbf{Z}(t) \mid N(t) > 0]$. These expected values are related as

$$\begin{aligned}\boldsymbol{\mu}(t) &= E[\mathbf{Z}(t) \mid N(t) = 0] \Pr\{N(t) = 0\} + E[\mathbf{Z}(t) \mid N(t) > 0] \Pr\{N(t) > 0\} \\ &= P_0 \mathbf{e}(t|\mathbf{z}_0, 0) + (1 - P_0)\boldsymbol{\mu}^0(t)\end{aligned} \tag{5.49}$$

Furthermore, let us introduce the processes and corresponding variables centralized with respect to the both expected values

$$\mathbf{Z}^0 = \mathbf{Z} - \boldsymbol{\mu} \quad , \quad \mathbf{z}^0 = \mathbf{z} - \boldsymbol{\mu} \tag{5.50}$$

$$\mathbf{Y}^0 = \mathbf{Z} - \boldsymbol{\mu}^0 \quad , \quad \mathbf{y}^0 = \mathbf{z} - \boldsymbol{\mu}^0 \tag{5.51}$$

Joint probability density function of the state vector $\mathbf{Z}(t)$ can be represented in form of the sum of the continuous and discrete parts as

$$\begin{aligned}f_{\{\mathbf{Z}\}}(\mathbf{z}, t) &= f_{\{\mathbf{Z}\}}(\mathbf{z}, t \mid N(t) = 0) \Pr\{N(t) = 0\} + f_{\{\mathbf{Z}\}}(\mathbf{z}, t \mid N(t) > 0) \Pr\{N(t) > 0\} \\ &= P_0 \prod_{i=1}^{n_s} \delta\Big(z_i - e_i(t|\mathbf{z}_0, 0)\Big) + (1 - P_0) f^0_{\mathbf{Z}}(\mathbf{y}^0, t)\end{aligned} \tag{5.52}$$

where n_s is the number of state variables.

In the first, discrete, part the Dirac delta spike $\delta(z_i - e_i)$ represents the finite probability of the system being at the deterministic state $z_i = e_i$ in the case of non-zero initial conditions, or at rest, i.e. $z_i = 0$, in the case of zero initial conditions. This probability is, of course, concentrated at the displacement $z_i = e_i$, or $z_i = 0$, respectively. The

second continuous part in (5.52), is specified by the conditional probability density function $f^0_{\{\mathbf{Z}\}}(\mathbf{y}^0, t)$ given that at least one impulse occurred.

Accordingly, let us define two types of the central moments: unconditional moments

$$\kappa_{i_1 \cdots i_n}(t) = E\left[\prod_{q=1}^{n} Z^0_{i_q}\right] = \int_{\mathcal{R}^{n_s}} \prod_{q=1}^{n} z^0_{i_q} f_{\{\mathbf{Z}\}}(\mathbf{z}, t) d\mathbf{z} \tag{5.53}$$

and the conditional ones

$$\kappa^0_{i_1 \cdots i_n}(t) = E_0\left[\prod_{q=1}^{n} Y^0_{i_q}\right] = \int_{\mathcal{R}^{n_s}} \prod_{q=1}^{n} y^0_{i_q} f^0_{\mathbf{Z}}(\mathbf{y}^0, t) d\mathbf{y}^0 \tag{5.54}$$

In the situations when the first discrete term in (5.52) is predominant, it can be predicted that the probability density function $f_{\{\mathbf{Z}\}}(\mathbf{z}, t)$ will be difficult to approximate by a truncated Gram-Charlier expansion. The reason is that the Dirac delta is difficult to approximate in terms of Hermite polynomials. Accordingly, closure approximations such as the cumulant neglect closure schemes (3.65), may not be valid for the unconditional central moments $\kappa_{i_1 \cdots i_n}$. Hence, the Gram-Charlier expansion can be applied to the continuous part in (5.52) only, i.e. for the conditional probability density $f^0_{\{\mathbf{Z}\}}(\mathbf{y}^0, t)$. This is equivalent to assuming the closure approximations for the conditional moments. In order to obtain the closure approximations for the unconditional moments they must be expressed in terms of the conditional moments, the latter will be subject to the closure approximations, and in turn the lower-order conditional moments will be expressed in terms of unconditional moments. Hence the relationships (identities) between the unconditional moments $\kappa_{i_1 \cdots i_n}$ and conditional moments $\kappa^0_{i_1 \cdots i_n}$ are required.

The density function $f_{\{\mathbf{Z}\}}(\mathbf{z}, t)$ given by (5.52) is inserted into (5.53), which yields

$$\kappa_{i_1 \cdots i_n} = P_0 \prod_{q=1}^{n} (e_{i_q} - \mu_{i_q}) + (1 - P_0) E_0\left[\prod_{q=1}^{n} Z^0_{i_q}\right] \tag{5.55}$$

Substituting, cf. (5.50) and (5.51)

$$Z^0_{i_q} = Y^0_{i_q} + \frac{P_0}{1 - P_0}\left(\mu_{i_q} - e_{i_q}\right) \tag{5.56}$$

into (5.55) one obtains

$$\begin{aligned} \kappa_{i_1 \cdots i_n} = {} & \frac{P_0[P_0^{n-1} - (P_0 - 1)^{n-1}]}{(1 - P_0)^{n-1}} \prod_{q=1}^{n}\left(\mu_{i_q} - e_{i_q}\right) + \\ & (1 - P_0) \sum_{r=0}^{n-2} \binom{n}{r} \left\{\kappa^0_{i_1 \cdots i_{n-r}} \prod_{q=n-r+1}^{n}\left(\mu_{i_q} - e_{i_q}\right)\right\}_s \left(\frac{P_0}{1 - P_0}\right)^r \end{aligned} \tag{5.57}$$

The inverse identity is obtained from (5.54) by substituting the conditional density function $f^0_{\{\mathbf{Z}\}}(\mathbf{y}^0, t)$ evaluated from (5.52). The result is

$$\kappa^0_{i_1\cdots i_n} = \frac{1}{1-P_0} E\left[\prod_{q=1}^{n} Y^0_{i_q}\right] - \frac{P_0}{1-P_0}\left(\frac{-1}{1-P_0}\right)^n \prod_{q=1}^{n}\left(\mu_{i_q} - e_{i_q}\right) \tag{5.58}$$

Upon inserting

$$Y^0_{i_q} = Z^0_{i_q} - \frac{P_0}{1-P_0}\left(\mu_{i_q} - e_{i_q}\right) \tag{5.59}$$

one obtains

$$\kappa^0_{i_1\cdots i_n} = \frac{1}{1-P_0}\sum_{r=0}^{n-2}\binom{n}{r}\left\{\kappa_{i_1\cdots i_{n-r}}\prod_{q=n-r+1}^{n}\left(\mu_{i_q} - e_{i_q}\right)\right\}_s\left(\frac{-P_0}{1-P_0}\right)^r +$$
$$\left(P_0^n - P_0\right)\left(\frac{-1}{1-P_0}\right)^n\prod_{q=1}^{n}\left(\mu_{i_q} - e_{i_q}\right) \tag{5.60}$$

For example, if the set of moment equations is to be closed at the 4th order moments, the identity (5.57) must be specified for the joint 5th order moments κ_{ijklm} and 6th order moments κ_{ijklmn}. Let us use the cumulant neglect closure for the conditional moments κ^0_{ijklm} and κ^0_{ijklmn}. The closure approximations are, cf. (3.64)

$$\kappa^0_{ijklm} = 10\left\{\kappa^0_{ij}\kappa^0_{klm}\right\}_s \tag{5.61}$$

$$\kappa^0_{ijklmn} = 15\left\{\kappa^0_{ij}\kappa^0_{klmn}\right\}_s + 10\left\{\kappa^0_{ijk}\kappa^0_{lmn}\right\}_s - 2\cdot 15\left\{\kappa^0_{ij}\kappa^0_{kl}\kappa^0_{mn}\right\}_s \tag{5.62}$$

Next, all the conditional moments entering the right-hand sides of (5.61) and (5.62) are expressed in terms of the unconditional moments with the help of identity (5.60), further (5.61) and (5.62) are inserted into (5.57). In the case of zero initial conditions the following closure approximations for the 5th and 6th order moments are finally arrived at

$$\kappa_{ijklm} = \frac{1}{1-P_0}10\left\{\kappa_{ij}\kappa_{klm}\right\}_s - \frac{P_0(1+P_0)}{(1-P_0)^2}10\left\{\mu_i\mu_j\kappa_{klm}\right\}_s -$$
$$\frac{2P_0}{(1-P_0)^2}15\left\{\mu_i\kappa_{jk}\kappa_{lm}\right\}_s + \frac{P_0}{1-P_0}5\left\{\mu_i\kappa_{jklm}\right\}_s +$$
$$\frac{P_0(1+4P_0+P_0^2)}{(1-P_0)^3}10\left\{\mu_i\mu_j\mu_k\kappa_{lm}\right\}_s +$$
$$\frac{P_0(1+11P_0+11P_0^2+P_0^3)}{(1-P_0)^4}\mu_i\mu_j\mu_k\mu_l\mu_m \tag{5.63}$$

$$\kappa_{ijklmn} = \frac{15}{1-P_0}\{\kappa_{ij}\kappa_{klmn}\}_s - \frac{15P_0}{1-P_0}\{\mu_i\mu_j\kappa_{klmn}\}_s +$$
$$\frac{10}{1-P_0}\{\kappa_{ijk}\kappa_{lmn}\}_s + \frac{P_0\left(1+P_0-2P_0^2\right)}{\left(1-P_0^3\right)}20\{\mu_i\mu_j\mu_k\kappa_{lmn}\}_s -$$
$$\frac{60P_0}{(1-P_0)^2}\{\mu_i\kappa_{jk}\kappa_{lmn}\}_s - \frac{2\cdot 15}{(1-P_0)^2}\{\kappa_{ij}\kappa_{kl}\kappa_{mn}\}_s + \frac{2P_0\cdot 45}{(1-P_0)^3}\{\mu_i\mu_j\kappa_{kl}\kappa_{mn}\}_s -$$
$$\frac{P_0(1+7P_0-5P_0^2-3P_0^3)}{(1-P_0)^4}15\{\mu_i\mu_j\mu_k\mu_l\kappa_{mn}\}_s +$$
$$\frac{P_0(1+20P_0-40P_0^3-5P_0^4)}{(1-P_0)^5}\mu_i\mu_j\mu_k\mu_l\mu_m\mu_n \tag{5.64}$$

In the steady state, i.e. as $t \to \infty$, it implies that $P_0 \to 0$, hence the discrete part of the density function (5.52) vanishes, and the usual cumulant-neglect closure approximations (3.65) are obtained from (5.63) and (5.64).

Example 5.1: Application of modified cumulant neglect closure approximations in the case of a system with cubic non-linearity: Duffing oscillator driven by a train of general pulses

Consider the dynamic response of a Duffing oscillator to a homogeneous ($\nu(t) = \text{const.} = \nu$) Poisson train of general pulses, governed by the equation (1.87), where the excitation is regarded as the response of an auxiliary linear filter to the train of Dirac delta impulses, as given by equation (1.85).

The state vector of an augmented system, consists of the displacement and velocity responses $Y(t)$, $\dot{Y}(t)$ of a non-linear oscillator and those of a linear filter $X(t)$, $\dot{X}(t)$, cf. (1.78), (1.79).

The centralized state variables are governed by the following stochastic equations

$$\left.\begin{aligned}
dZ_1^0 &= B_{12}Z_2^0 dt \\
dZ_2^0 &= \left(A_2 + B_{21}Z_1^0 + B_{22}Z_2^0 + B_{23}Z_3^0 + C_{211}{Z_1^0}^2 + D_{2111}{Z_1^0}^3\right)dt \\
dZ_3^0 &= B_{34}Z_4^0 dt \\
dZ_4^0 &= (B_{43}Z_3^0 + B_{44}Z_4^0 - b_4\nu(t)E[P])dt + b_4\int_{\mathcal{P}} pM(dt,t,dp,p)
\end{aligned}\right\} \tag{5.65}$$

where

$$\left.\begin{aligned}
&B_{12} = 1,\ A_2 = \varepsilon\omega_0^2\left(\mu_{111} + 3\mu_{11}m_1\right) \\
&B_{21} = -\omega_0^2\left(1+3\varepsilon\mu_1^2\right),\ B_{22} = -2\zeta\omega_0 \\
&B_{34} = 1,\ B_{43} = -\omega_f^2\ ,\ B_{44} = -2\zeta_f\omega_f \\
&C_{211} = -3\varepsilon\omega_0^2 m_1\ ,\ D_{2111} = -\varepsilon\omega_0^2
\end{aligned}\right\} \tag{5.66}$$

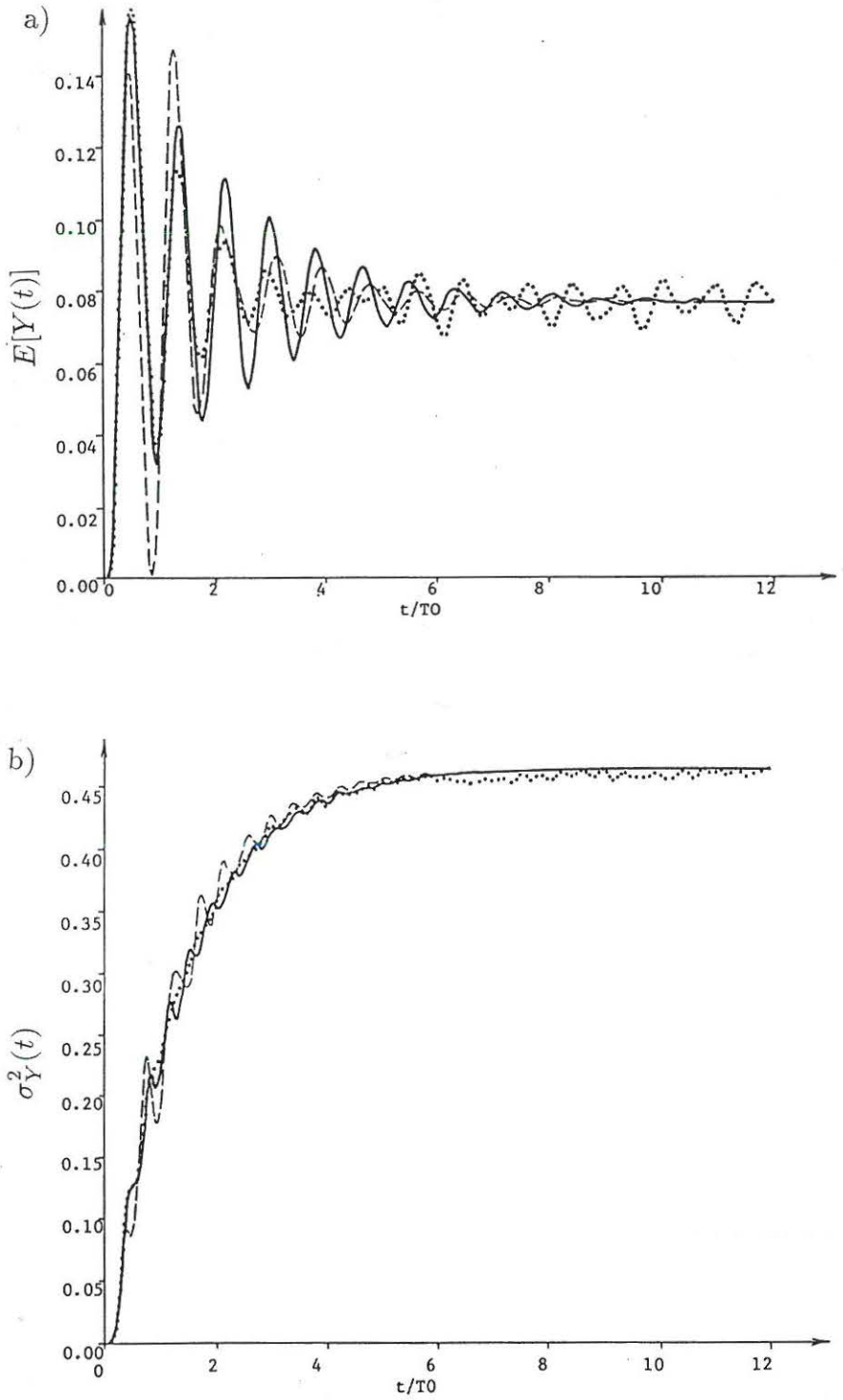

Fig. 5.1. Stochastic response of a Duffing oscillator driven by a Poissonian train of general pulses with $\nu = 0.1\omega_0$. a) Mean value function. b) Variance function. – – – ordinary closure technique, _____ modified closure technique, · · · · · Monte Carlo simulation.

and ζ_f and ω_f are the filter damping ratio and natural frequency, respectively. If the damping ratio ζ_f is close to unity, the impulse response function of the filter $h(t-t_i)$ decays very fast and becomes practically a single pulse (damped sine half-wave) with duration $\Delta = T_f/2$, where $T_f = 2\pi/\omega_f\sqrt{1-\zeta_f^2}$ is the natural period of the filter.

In order to perform the example computations the following data has been assumed $\varepsilon = 0.5, \zeta = 0.05, \nu = 0.1\omega_0$ and $\nu = 0.05\omega_0$. Impulses magnitudes have been assumed as non-zero mean, Rayleigh-distributed random variables. The parameter σ_P of a Rayleigh distribution is chosen in such a way that in both cases $\nu = 0.1\omega_0$ and $\nu = 0.05\omega_0$ the variance of the stationary response of a corresponding linear oscillator ($\varepsilon = 0$) has a unit value. The following data has been assumed for the filter: $\omega_f = 2.4\omega_0$, $\zeta_f = 0.95$. Then $\Delta = \pi/\omega_f\sqrt{1-\zeta_f^2} = \frac{2}{3}T_0$. Hence the pulse duration is comparable with the natural period of a linear oscillator.

The set of moment equations has been truncated at fourth-order moments level. In the case $\nu = 0.1\omega_0$ the closure approximations (3.65) resulting from the ordinary cumulant neglect closure and the modified closure approximations (5.63) and (5.64) have been applied, both based on neglecting the 5th and 6th order cumulants. The mean value and variance responses are shown in fig. 5.1 for $\nu = 0.1\omega_0$, evaluated with the help of ordinary closure approximations are accurate enough, yet the use of modified closure approximations gives even better predictions. In the case $\nu = 0.05\omega_0$, shown in fig. 5.2, the ordinary cumulant neglect closure approximations provides completely erroneous results. The predicted response variance becomes negative. However, the modified closure approximations lead to very good estimates of both the mean value and the variance of the response.

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5.5 Example problems

5.1. Derive the equations for first and second-order moments of a linear oscillator. Evaluate the stationary response mean value and variance.

5.2. Derive the necessary conditions on $c_2(z_1, z_2)$ and $b_2(z_1, z_2)$ in order that (5.14) may be a solution of (5.11).

CHAPTER 6

NON-MARKOV RESPONSE PROBLEMS REDUCIBLE TO MARKOV PROBLEMS

6.1 Dynamic response of non-linear systems to Erlang renewal impulse process excitations: Markov approach

6.1.1 Governing stochastic integro-differential equations for Erlang driven non-Markov response problems.

Consider a general multi-degree-of-freedom non-linear dynamical system under a random train of impulses driven by a renewal point process, The state vector of the system $\mathbf{Z}_1(t)$ is governed by the set of equations

$$\frac{d}{dt}\mathbf{Z}_1(t) = \mathbf{c}_1\big(\mathbf{Z}_1(t),t\big) + \mathbf{b}_1\big(\mathbf{Z}_1(t),t\big)\sum_{i=1}^{R(t)} P_{i,R}\delta\big(t-t_{i,R}\big) =$$

$$\mathbf{c}_1\big(\mathbf{Z}_1(t),t\big) + \mathbf{b}_1\big(\mathbf{Z}_1(t),t\big)\sum_{i=1}^{\infty} P_{i,R}\delta\big(t-t_{i,R}\big)\hat{\mathbf{1}}\big(t-t_{i,R}\big) \tag{6.1}$$

where $\hat{\mathbf{1}}(\cdot)$ is the indicator function defined by (1.16).

The occurrence times $t_{i,R}$ of Dirac delta impulses are distributed according to an ordinary renewal counting process $\{R(t), t \in [0,\infty[\}, \Pr\{R(0)=0\}=1$. The mark variables $P_{i,R}$ are assumed to be independent, identically distributed, random variables, independent of the occurrence times $t_{i,R}$ and having the distribution as a random variable P. It is obvious that since the renewal process is not a process with independent increments, so the state vector does not obey the Markov property.

Let us confine the considerations to the class of Erlang renewal processes, i.e. the ones for which the interarrival times $T_n = t_n - t_{n-1}$ are independent, gamma distributed random variables, i.e. $T_n \sim \mathrm{G}(k-1,\nu)$ with the probability density function given by (4.42). Recall that the events driven by an Erlang process with parameter k can be regarded as every kth Poisson events taken out of a stationary Poisson process with the mean arrival rate ν, cf. (4.42).

The idea is to recast the renewal-driven impulse process, or the excitation term of the equation (6.1) in such a way as to obtain a non-zero impulse magnitude for kth, $2k$th, $3k$th ... i.e. every kth Poisson event and zero magnitudes for all other Poisson events. The Poisson counting process $\{N(t), t \in [0,\infty[\}$ is defined as the number of events in the time interval $[0,t[$, hence the additional assumption is made: $\Pr\{N(0)=0\}=1$.

Hence the governing stochastic equations should be converted to the form of

$$\frac{d}{dt}\mathbf{Z}_1(t) = \mathbf{c}_1\big(\mathbf{Z}_1(t),t\big) + \mathbf{b}_1\big(\mathbf{Z}_1(t),t\big)\sum_{i=1}^{N(t)} \rho\big(N(t_i)\big)P_i\delta\left(t-t_i\right) =$$

$$\mathbf{c}_1\big(\mathbf{Z}_1(t),t\big) + \mathbf{b}_1\big(\mathbf{Z}_1(t),t\big)\sum_{i=1}^{\infty} \rho\big(N(t_i)\big)P_i\delta\left(t-t_i\right)\hat{\mathbf{1}}\big(t-t_i\big) \tag{6.2}$$

where $\rho\big(N(t_i)\big)$ is the required transformation of the Poisson counting process $N(t_i)$, such that $\rho\big(N(t_i)\big) = 1$ for every kth Poisson event and $\rho\big(N(t_i)\big) = 0$ for all other Poisson events, $N(t_i)$ being the number of past Poisson events, not including the one which occurs at the time t_i.

Next the equations (6.2) are integrated over time, which yields

$$\mathbf{Z}_1(t) = \mathbf{Z}_{1,0} + \int_0^t \mathbf{c}_1\big(\mathbf{Z}_1(\tau),\tau\big)d\tau + \int_0^t \int_{\mathcal{P}} \mathbf{b}_1\big(\mathbf{Z}_1(\tau),\tau\big)\rho\big(N(\tau)\big)pM(d\tau,\tau,dp,p) \quad (6.3)$$

where

$$\int_0^t \int_{\mathcal{P}} \mathbf{b}_1\big(\mathbf{Z}_1(\tau),\tau\big)\rho\big(N(\tau)\big)pM(d\tau,\tau,dp,p) = \sum_{i=1}^{\infty} \mathbf{b}_1\big(\mathbf{Z}_1(t_i),t_i\big)\rho\big(N(t_i)\big)P_i\hat{\mathbf{1}}(t-t_i) \quad (6.4)$$

for $N(t) \geq 1$, $M(d\tau,\tau,dp,p)$ is the random Poisson measure defined by (1.44)–(1.49) and $\mathbf{Z}_{1,0}$ is the vector of possible random initial conditions. In what follows let us assume the zero initial conditions $\mathbf{Z}_{1,0} \equiv 0$.

The stochastic integro-differential equations governing the system state vector can then be written as

$$d\mathbf{Z}_1(t) = \mathbf{c}_1\big(\mathbf{Z}_1(t),t\big)dt + \mathbf{b}_1\big(\mathbf{Z}_1(t),t\big)\rho\big(N(t)\big)\int_{\mathcal{P}} pM(dt,t,dp,p) \quad (6.5)$$

6.1.2 Converting the non-Markov problem to a Markov one by suitable recasting of the excitation process

The transformation satisfying the required property is found to be

$$\rho\big(N(t)\big) = \frac{1}{k}\sum_{j=0}^{k-1} \exp\left(i2\pi\frac{j(N(t)+1)}{k}\right) = \frac{1}{k}\sum_{j=0}^{k-1} U_j \quad (6.6)$$

$$U_j = \exp\left(i2\pi\frac{j(N(t)+1)}{k}\right) \quad (6.7)$$

For $N(t) = 0,1,2,\ldots,k-1$ the first term at the right-hand side of (6.6) is the Finite Fourier Transform of the k-dimensional sequence $\{0,0,\ldots,0,1\}$. From the periodicity properties it follows that

$$\rho\big(N(t)\big) = \begin{cases} 1 & , \quad N(t) = k-1,\ 2k-1,\ 3k-1,\ldots \\ 0 & , \quad \text{else} \end{cases} \quad (6.8)$$

which means that $\rho\big(N(t)\big) = 1$ as every kth Poissonian impulse arrives.

Seeing that $U_j = U^*_{k-j}$, where * denotes the complex conjugate the right-hand side of (6.6) can be evaluated as

$$\rho\big(N(t)\big) = \begin{cases} \frac{1}{k}\big(1 + U_1 + \ldots + U_{k_0-1} + U^*_{k_0-1} + \ldots + U^*_1\big) & , \quad k \text{ odd} \\ \frac{1}{k}\big(1 + U_1 + \ldots + U_{k_0-1} + U_{k_0} + U^*_{k_0-1} + \ldots + U^*_1\big) & , \quad k \text{ even} \end{cases}$$

$$\begin{cases} \frac{1}{k}\left(1 + 2\sum_{j=1}^{k_0-1} C_j\right) & , \quad k \text{ odd} \\ \frac{1}{k}\left(1 + 2\sum_{j=1}^{k_0-1} C_j + C_{k_0}\right) & , \quad k \text{ even} \end{cases} \tag{6.9}$$

where

$$k_0 = \left[\frac{k+1}{2}\right] \tag{6.10}$$

$[\cdot]$ being the integer part and

$$C_j = \Re(U_j) = \cos\left(2\pi\frac{j(N(t)+1)}{k}\right), \quad j = 1, 2, \ldots, k_0 - 1 \tag{6.11}$$

$$S_j = \Im(U_j) = \sin\left(2\pi\frac{j(N(t)+1)}{k}\right), \quad j = 1, 2, \ldots, k_0 - 1 \tag{6.12}$$

$$C_{k_0} = \exp\big(i\pi(N(t)+1)\big) = (-1)^{N(t)+1} = -\cos\big(\pi N(t)\big), \quad k \text{ even} \tag{6.13}$$

These transformations of a Poisson counting process $N(t)$ will be regarded as additional state variables.

The stochastic equations for these variables are obtained from

$$\begin{aligned} dU_j(t) &= U_j(t+dt) - U_j(t) = \exp\left(i2\pi\frac{j\big(N(t+dt)+1\big)}{k}\right) - \exp\left(i2\pi\frac{j\big(N(t)+1\big)}{k}\right) \\ &= \exp\left(i2\pi\frac{j\big(N(t)+dN(t)+1\big)}{k}\right) - \exp\left(i2\pi\frac{j\big(N(t)+1\big)}{k}\right) \\ &= U_j(t)\left(\exp\left(i2\pi\frac{j}{k}dN(t)\right) - 1\right) \end{aligned} \tag{6.14}$$

Since the increment $dN(t)$ of the regular counting process can only attain the values 0 or 1 with non-vanishing probabilities, (6.14) can be written as

$$dU_j(t) = U_j(t)\left(\exp\left(i2\pi\frac{j}{k}\right) - 1\right)dN(t) \quad , \quad j = 1, 2, \ldots, k_0 \tag{6.15}$$

The equivalence of (6.14) and (6.15) follows from the fact that the right-hand sides give the same result for both $dN(t) = 0$ and $dN(t) = 1$.

Specifically, the equations for the real and imaginary parts become

$$dC_j = \left(C_j\left(\cos\left(2\pi\frac{j}{k}\right) - 1\right) - S_j \sin\left(2\pi\frac{j}{k}\right)\right)dN(t) \quad , \quad j = 1, 2, \ldots, k_0 - 1 \tag{6.16}$$

$$dS_j = \left(C_j \sin\left(2\pi\frac{j}{k}\right) + S_j\left(\cos\left(2\pi\frac{j}{k}\right) - 1\right)\right)dN(t) \quad , \quad j = 1, 2, \ldots, k_0 - 1 \tag{6.17}$$

$$dC_{k_0} = -2C_{k_0}dN(t) \quad , \quad k \text{ even} \tag{6.18}$$

It is seen that new state variables have been introduced: C_j, S_j and for k even also C_{k_0}. The state vector augmented by these new variables is governed by the stochastic equations

$$d\mathbf{Z}(t) = \mathbf{c}\big(\mathbf{Z}(t), t\big)dt + \int_{\mathcal{P}} \mathbf{b}\big(\mathbf{Z}(t), t, p\big)M(dt, t, dp, p) \tag{6.19}$$

$$\mathbf{Z}(t) = \begin{bmatrix} \mathbf{Z}_1(t) \\ \mathbf{Z}_2(t) \end{bmatrix}, \ \mathbf{c}\big(\mathbf{Z}(t), t\big) = \begin{bmatrix} \mathbf{c}_1\big(\mathbf{Z}_1(t), t\big) \\ \mathbf{0} \end{bmatrix}, \ \mathbf{b}\big(\mathbf{Z}(t), t, p\big) = \begin{bmatrix} \rho\big(N(t)\big)\mathbf{b}_1\big(\mathbf{Z}_1(t), t\big)p \\ \mathbf{b}_2\big(\mathbf{Z}_2(t)\big) \end{bmatrix} \tag{6.20}$$

where for k even

$$\mathbf{Z}_2(t) = \begin{bmatrix} C_1 \\ S_1 \\ C_2 \\ S_2 \\ \vdots \\ C_{k_0-1} \\ S_{k_0-1} \\ C_{k_0} \end{bmatrix}, \ \mathbf{b}_2\big(\mathbf{Z}_2(t)\big) = \begin{bmatrix} C_1\left(\cos\left(2\pi\frac{1}{k}\right) - 1\right) - S_1 \sin\left(2\pi\frac{1}{k}\right) \\ C_1 \sin\left(2\pi\frac{1}{k}\right) + S_1\left(\cos\left(2\pi\frac{1}{k}\right) - 1\right) \\ C_2\left(\cos\left(2\pi\frac{2}{k}\right) - 1\right) + S_2 \sin\left(2\pi\frac{2}{k}\right) \\ C_2 \sin\left(2\pi\frac{2}{k}\right) + S_2\left(\cos\left(2\pi\frac{2}{k}\right) - 1\right) \\ \vdots \\ C_{k_0-1}\left(\cos\left(2\pi\frac{k_0-1}{k}\right) - 1\right) - S_{k_0-1} \sin\left(2\pi\frac{k_0-1}{k}\right) \\ C_{k_0-1} \sin\left(2\pi\frac{k_0-1}{k}\right) + S_{k_0-1}\left(\cos\left(2\pi\frac{k_0-1}{k}\right) - 1\right) \\ -2C_{k_0} \end{bmatrix} \tag{6.21}$$

and for k odd

$$\mathbf{Z}_2(t) = \begin{bmatrix} C_1 \\ S_1 \\ C_2 \\ S_2 \\ \vdots \\ C_{k_0-1} \\ S_{k_0-1} \end{bmatrix}, \quad \mathbf{b}_2\big(\mathbf{Z}_2(t)\big) = \begin{bmatrix} C_1\Big(\cos\big(2\pi\frac{1}{k}\big)-1\Big) - S_1\sin\big(2\pi\frac{1}{k}\big) \\ C_1\sin\big(2\pi\frac{1}{k}\big) + S_1\Big(\cos\big(2\pi\frac{1}{k}\big)-1\Big) \\ C_2\Big(\cos\big(2\pi\frac{2}{k}\big)-1\Big) + S_2\sin\big(2\pi\frac{2}{k}\big) \\ C_2\sin\big(2\pi\frac{2}{k}\big) + S_2\Big(\cos\big(2\pi\frac{2}{k}\big)-1\Big) \\ \vdots \\ C_{k_0-1}\Big(\cos\big(2\pi\frac{k_0-1}{k}\big)-1\Big) - S_{k_0-1}\sin\big(2\pi\frac{k_0-1}{k}\big) \\ C_{k_0-1}\sin\big(2\pi\frac{k_0-1}{k}\big) + S_{k_0-1}\Big(\cos\big(2\pi\frac{k_0-1}{k}\big)-1\Big) \end{bmatrix} \tag{6.22}$$

If the excitation is purely external, i.e. $\mathbf{b}_1\big(\mathbf{Z}_1(t),t\big) \equiv \mathbf{b}_1$ (constant), then

$$\mathbf{b}\big(\mathbf{Z}(t),p\big) = \mathbf{b}_0 p + \mathbf{B}(p)\mathbf{Z}(t) \tag{6.23}$$

where $\mathbf{b}_0$ is a constant vector and $\mathbf{B}$ is a constant (state independent) matrix. Hence, even though the excitation is purely external due to the introduction of auxiliary state variables, in the formulation for the augmented state vector, the vector $\mathbf{b} = \mathbf{b}\big(\mathbf{Z}(t)\big)$ becomes state dependent and is a linear form in the state variables.

The state vector $\mathbf{Z}(t)$, augmented by additional, auxiliary state variables, as governed by equation (6.19) is driven by a Poisson process, and hence it is a *Markov vector process.*

6.1.3 Differential equations for moments

Equations for the mean values $\mu_i(t) = E\left[Z_i(t)\right]$ are obtained by direct averaging of the governing stochastic equations (6.19), cf. (5.26)

$$\dot{\boldsymbol{\mu}}(t) = E\Big[\mathbf{c}\big(\mathbf{Z}(t),t\big)\Big] + \nu \int_{\mathcal{P}} E\Big[\mathbf{b}\big(\mathbf{Z}(t),t,p\big)\Big] f_P(p)dp =$$

$$E\Big[\mathbf{c}\big(\mathbf{Z}(t),t\big)\Big] + \nu E\Big[\mathbf{b}\big(\mathbf{Z}(t),t,P\big)\Big] \tag{6.24}$$

where P is the random magnitude of the impulse.

In the case of a purely external excitation at the right-hand sides of equations for mean values the renewal density appears, see section 4.1.3, which can be shown in the following way. In agreement with the representation (6.2) the increment $dR(t)$ of the renewal counting process $R(t)$ during the infinitesimal time interval $[t, t+dt[$ is

$$\rho\big(N(t)\big)dN(t) = dR(t) \tag{6.25}$$

Notice that $\rho(N(t))$ is affected by the increments in the interval $[0,t[$ and is independent of the increment $dN(t) = N(t+dt) - N(t)$ in the interval $[t,t+dt[$. Therefore the expectation splits as follows

$$E[\rho(N(t))dN(t)] = E[\rho(N(t))]E[dN(t)] = \nu E[\rho(N(t))]dt \tag{6.26}$$

Further, by definition, cf. (4.6) and (4.27)

$$E[dR(t)] = h_o(t)dt \tag{6.27}$$

where $h_o(t)$ is the ordinary renewal density.

Upon taking the expectation of both sides of (6.25) the following result is obtained

$$E[\rho(N(t))dN(t)] = \nu E[\rho(N(t))]dt = h_o(t)dt \tag{6.28}$$

Equations for zero-mean state variables (responses) $\mathbf{Z}^0(t) = \mathbf{Z}(t) - \boldsymbol{\mu}(t)$ are obtained as

$$d\mathbf{Z}^0(t) = \mathbf{c}^0(\mathbf{Z}^0(t),t)dt + \int_{\mathcal{P}} \mathbf{b}(\mathbf{Z}(t),t,p)M(dt,t,dp,p) \tag{6.29}$$

where

$$\mathbf{c}^0(\mathbf{Z}^0(t),t) = \mathbf{c}(\mathbf{Z}^0(t)+\boldsymbol{\mu}(t),t) - E\Big[\mathbf{c}(\mathbf{Z}^0(t)+\boldsymbol{\mu}(t),t)\Big] - \nu E\Big[\mathbf{b}(\mathbf{Z}(t),t,P)\Big] \tag{6.30}$$

The equations for the joint central moments are obtained from the general equation (5.30).

The initial conditions associated with the equations for moments are non-zero, since some of the auxiliary state variables C_j, S_j and C_{k_0} as defined by equations (6.11)–(6.13), and hence their mean values assume non-zero initial values.

For example in the cases of $k=2$, $k=3$ and $k=4$ the required transformations of the Poisson counting process $N(t)$, such that $\rho(N(t)) = 1$ for every 2nd, 3rd and 4th Poisson event and $\rho(N(t)) = 0$ for all other Poisson events are, respectively

$$\rho(N(t)) = \frac{1}{2}(1+C_{k_0}) = \frac{1}{2}(1-(-1)^{N(t)}) = \frac{1}{2}\Big(1-\cos(\pi N(t))\Big) \tag{6.31}$$

$$\rho(N(t)) = \frac{1}{3}(1+2C_1) = \frac{1}{3}\Big(1-\sqrt{3}\sin\Big(\frac{2}{3}\pi N(t)\Big) - \cos\Big(\frac{2}{3}\pi N(t)\Big)\Big) \tag{6.32}$$

$$\rho(N(t)) = \frac{1}{4}(1+2C_1+C_{k_0}) = \frac{1}{4}\Big(1-2\sin\Big(\frac{1}{2}\pi N(t)\Big) - \cos\Big(\pi N(t)\Big)\Big) \tag{6.33}$$

Example 6.1: Response of a Duffing oscillator to an Erlang renewal process driven impulse process excitation with k = 2, k = 3, and k = 4

To illustrate the moment equations technique developed, consider a Duffing oscillator with displacement $Y(t)$ and velocity $\dot{Y}(t)$, where $\mathbf{Z}_1(t)$, $\mathbf{c}_1(\mathbf{Z}_1(t),t)$ and $\mathbf{b}_1(\mathbf{Z}_1(t),t)$ of (6.1) are given by

$$\mathbf{Z}_1(t) = \begin{bmatrix} Y(t) \\ \dot{Y}(t) \end{bmatrix}, \quad \mathbf{c}_1(\mathbf{Z}_1(t),t) = \begin{bmatrix} \dot{Y}(t) \\ -2\zeta\omega_0\dot{Y}(t) - \omega_0^2 Y(t) - \varepsilon\omega_0^2 Y^3(t) \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{6.34}$$

where ζ is the damping ratio, ω_0 is the circular eigenfrequency of the corresponding linear oscillator and ε is the non-linearity parameter.

It appears that the stationary mean response of a linear oscillator to a renewal impulse process is obtained as

$$E[Y] = \mu_1 = \frac{\nu}{k}\frac{E[P]}{\omega_0^2} \tag{6.35}$$

which is exactly the same as the stationary mean response of a linear oscillator to Poisson distributed impulses with the mean rate $\frac{\nu}{k}$.

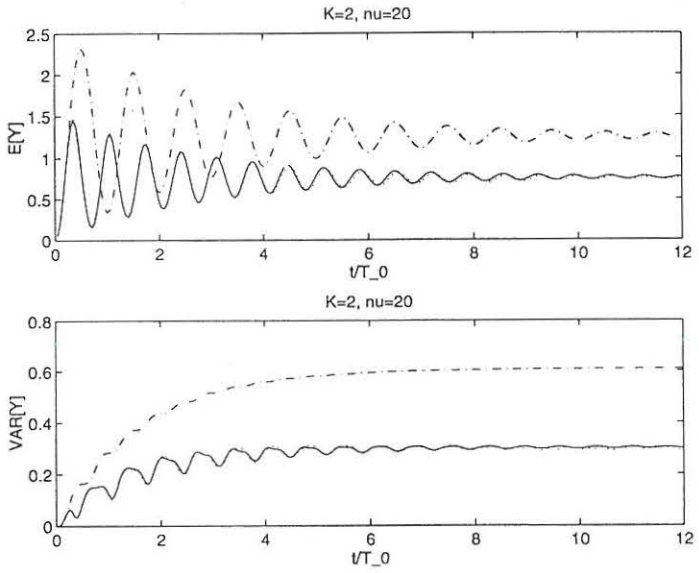

Fig. 6.1. Duffing oscillator subjected to Erlang renewal process. Mean value $E[Y]$ and variance $Var[Y]$ in the case $k = 2, \nu = 20\omega_0$. ——— : non-linear response, analytical results. · · · · · : non-linear response, simulated results. – · – · – · : linear response.

In order for the different cases of the renewal impulse proceses to be comparable the value of the parameter ν, for different k, is assumed in such a way as to maintain the same ratio $\frac{\nu}{k}$ and hence the same mean arrival rate $\frac{\nu}{k}$ of the comparative, or corresponding, Poisson impulse process. The variance of the stationary response of a linear oscillator to Poissonian impulses with the mean arrival rate $\frac{\nu}{k}$ is, given by

$$\sigma_Y^2 = \kappa_{11} = \frac{\nu}{k}\frac{E[P^2]}{4\zeta\omega_0^3} \tag{6.36}$$

Hence in all considered cases the stationary variance of the response to a comparative Poisson impulse process is the same. Moreover the data for the random variable P is assumed in such a way that σ_Y^2 as given by (6.36) has unit value.

The data assumed for the Duffing oscillator is: $\omega_0 = 1s^{-1}$, $\zeta = 0.05$, $\varepsilon = 0.5$. With this value of the parameter ε the non-linearity should be regarded as quite strong, since the mean value and the variance of the response of the Duffing oscillator to the Poisson train of impulses are then substantially different from the statistics of the response of a linear oscillator, cf. [6.3], [6.4]. The values of ν for the cases $k = 2$, $k = 3$ and $k = 4$ are assumed, respectively as $\nu = 20\omega_0$, $\nu = 30\omega_0$ and $\nu = 40\omega_0$ hence in all the considered cases the mean arrival rate of a comparative Poisson process is $\frac{\nu}{k} = 10\omega_0$. The impulses magnitudes are assumed to be Rayleigh-distributed random variables, $P \sim R(\sigma_P^2)$, with $\sigma_P = 0.1s^{-1}$.

The equations for moments up to and including the fourth order moments, have been derived. Since the drift terms $c_i^0(\mathbf{Z}_1^0(t))$ are the cubic forms in the state variables, the 5th order moments appear in the equations for 3rd order moments, see (5.32), and 5th and 6th order moments appear in the equations for 4th order moments, see (5.33). These redundant moments have been evaluated with the help of the modified cumulant neglect closure scheme devised by the authors for Poisson driven pulse problems [6.4].

To verify the approximate analytical results, the Monte Carlo simulations have been performed. The simulated results were obtained based on averaging over an ensemble of 32000 independent response sample curves, each obtained by numerical integration of governing equation of motion (6.1).

Both the equations for moments and the governing equation of motion (in the simulation technique) have been solved with the help of 4th order Runge-Kutta numerical integration technique. In order to obtain the excitation sample functions, the interarrival times of impulses are generated with the help of negative exponential distributed random variables. In each case of a renewal process the interarrival times are constructed as a sum of k auxiliary negative exponential distributed variates. Next the impulses magnitudes generated from a Rayleigh distribution are assigned to the impulses arrival times. The response sample curves are obtained by numerical integration of the homogeneous governing equation of motion (6.1) between the impulses arrival times, whereas at each arrival time the velocity is increased by a jump, which gives the updated initial condition for the next interarrival time interval.

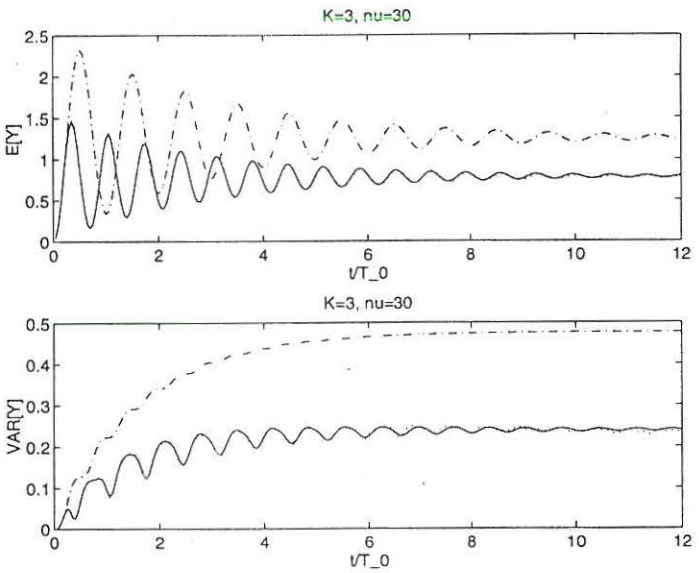

Fig. 6.2. Duffing oscillator subjected to Erlang renewal process. Mean value $E[Y]$ and variance $Var[Y]$ in the case $k = 3, \nu = 30\omega_0$. ——— : non-linear response, analytical results. – – – : non-linear simulated response. – · – · – · : linear response.

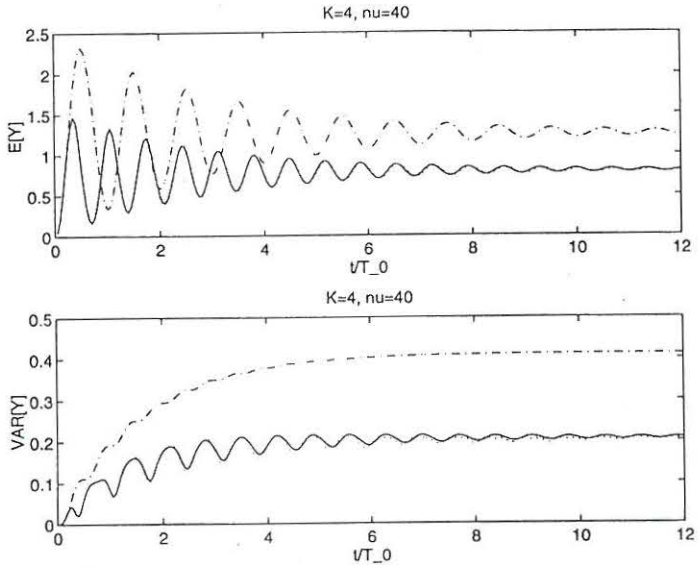

Fig. 6.3. Duffing oscillator subjected to Erlang renewal process. Mean value $E[Y]$ and variance $Var[Y]$ in the case $k = 4, \nu = 40\omega_0$. ——— : non-linear response, analytical results. – – – : non-linear simulated response. – · – · – · : linear response.

The results for the mean value $\mu_1(t) = E[Y]$ and variance $\kappa_{11}(t) = \sigma_Y^2(t) = Var[Y]$ evaluated for the cases $k = 2$, $k = 3$ and $k = 4$ are shown in Figs. 1-3, respectively, versus relative time t/T_0, where T_0 is the eigenperiod of the corresponding linear oscillator. Agreement of the approximate analytical results with the simulated ones is very good.

The mean responses of the Duffing oscillator are in all three cases practically the same. It should be noted that the mean response of a Duffing oscillator to a renewal impulses with $k = 2$ is only slightly different (larger) than the response to a comparative Poisson impulse process, cf. [6.3].

The variance of the response, both in the case of a linear and of a Duffing oscillator decreases as the parameter k increases (note that the variance of the stationary response to a comparative Poisson impulse process with the mean rate $\frac{\nu}{k}$ is in all the cases equal to 1).

6.2 Markov approach in a more general case of renewal impulses

6.2.1 Statement of the problem for arbitrary renewal impulse processes

Consider a general multi-degree-of-freedom non-linear dynamical system under a random train of impulses driven by a renewal point process. The structural state vector, $\mathbf{Z}_1(t)$, consisting of the generalized displacements and velocities is governed by the set of equations of motion (6.1).

Consider an arbitrary regular renewal counting process $R(t)$. Assume that its increments can be expressed as

$$dR(t) = \rho\big(N(t)\big)\,dN(t) \tag{6.37}$$

where $N(t)$ is a homogeneous Poisson counting process and $\rho\big(N(t)\big)$ is a suitable zero-memory transformation to be determined later. The nth degree product density $f_n(t_1,\dots,t_n)$ of the point process $R(t)$ is defined as cf. Srinivasan [6.6] and (4.2)

$$f_n(t_1,\dots,t_n)dt_1\dots dt_n =$$

$$E\big[dR(t_1)\cdots dR(t_n)\big] = E\big[\rho\big(N(t_1)\big)dN(t_1)\cdots\rho\big(N(t_n)\big)dN(t_n)\big] \tag{6.38}$$

It may be shown that by suitable splitting of expectations the expression (6.38) for any renewal process may be recast as

$$f_n(t_1,\dots,t_n)dt_1\cdots dt_n = E\big[\rho\big(N(t_1)\big)\big]\cdots E\big[\rho\big(N(t_n-t_{n-1})\big)\big]\nu^n dt_1\cdots dt_n \tag{6.39}$$

Seeing that $\nu E\big[\rho\big(N(t)\big)\big] = h_o(t)$, which is the ordinary renewal density, the known result for the nth degree product density of the ordinary renewal process is obtained, cf. (4.37)

$$f_n(t_1,\dots,t_n) = h_o(t_1)h_o(t_2-t_1)\cdots h_o(t_n-t_{n-1}) \tag{6.40}$$

Let us introduce

$$Y_j(t) = \rho\big(N(t)+j-1\big) \quad , \quad j = 1,2,\dots,k \tag{6.41}$$

as new auxiliary state variables. The stochastic differential equations governing the time evolution of these state variables can be written as

$$dY_j(t) = \rho\big((N(t))+j-1+dN(t)\big) - \rho\big(N(t)+j-1\big) =$$

$$\Big(\rho\big((N(t))+j\big) - \rho\big(N(t)+j-1\big)\Big)dN(t) \quad \Rightarrow$$

$$\left.\begin{aligned} dY_1(t) &= \Big(Y_2(t)-Y_1(t)\Big)dN(t)\\ dY_2(t) &= \Big(Y_3(t)-Y_2(t)\Big)dN(t)\\ &\vdots\\ dY_{k-1}(t) &= \Big(Y_k(t)-Y_{k-1}(t)\Big)dN(t) \end{aligned}\right\} \tag{6.42}$$

Validity of the statements given in the first part of eqs. (6.42) is proved by showing that it is valid for either of $dN(t) = 0$ or $dN(t) = 1$. The hierarchy of stochastic differential equations (6.42) cannot be closed unless $Y_k(t)$ can be expressed in terms of the previous auxiliary state variables. The following linear dependency is adopted, which can be shown to be valid in the case of an ordinary Erlang process

$$Y_k(t) = -\frac{1}{a_k}\Big(a_0 + a_1Y_1(t) + \cdots + a_{k-1}Y_{k-1}(t)\Big) \tag{6.43}$$

where $a_j \in R$. Equations (6.42) then attain the form

$$\left.\begin{aligned} dY_1(t) &= \Big(Y_2(t) - Y_1(t)\Big)dN(t) \\ dY_2(t) &= \Big(Y_3(t) - Y_2(t)\Big)dN(t) \\ &\vdots \\ dY_{k-1}(t) &= -\left(\frac{a_0}{a_k} + \frac{a_1}{a_k}Y_1(t) + \cdots + \frac{a_{k-2}}{a_k}Y_{k-2}(t) + \Big(1 + \frac{a_{k-1}}{a_k}\Big)Y_{k-1}(t)\right)dN(t) \end{aligned}\right\} \quad (6.44)$$

Equation (6.43) implies that $\rho\big(N(t)\big)$ must fulfill the difference equation

$$a_0 + \sum_{j=1}^{k} a_j \rho\big(N(t) + j - 1\big) = 0 \quad (6.45)$$

The solution of (6.45) is

$$\rho\big(N(t)\big) = b_0 + \sum_{j=1}^{k-1} b_j \lambda_j^{N(t)} = \sum_{j=0}^{k-1} b_j \lambda_j^{N(t)} \quad (6.46)$$

where $\lambda_0 = 1$ and

$$b_0 = -\frac{\frac{a_0}{a_k}}{\frac{a_1}{a_k} + \cdots + \frac{a_{k-1}}{a_k} + 1} \quad (6.47)$$

$b_j \in C,\ j = 1, \ldots, k-1$ are arbitrary constants and $\lambda_j \in C,\ j = 1, \ldots, k-1$ denote the solutions of the characteristic equation

$$\lambda^{k-1} + \frac{a_{k-1}}{a_k}\lambda^{k-2} + \cdots + \frac{a_2}{a_k}\lambda + \frac{a_1}{a_k} = 0 \quad (6.48)$$

If $|\ \lambda_j\ | \neq 1$, the corresponding term in (6.46) either extincts or explodes. Hence if the point process is assumed to be homogeneous, it is necessary that the eigenvalues all have the magnitude equal to 1, so

$$\lambda_j = \exp\Big(i\gamma_j\Big), \quad \gamma_j \in R/\{0\}, \quad j = 1, 2, \ldots, k-1 \quad (6.49)$$

The product density becomes, cf. (6.38) and (6.46)

$$f_n(t_1, \ldots, t_n)dt_1 \cdots dt_n =$$

$$\sum_{j_1=0}^{k-1} \cdots \sum_{j_n=0}^{k-1} b_{j_1} \cdots b_{j_n} E\Big[\lambda_{j_1}^{N(t_1)} \cdots \lambda_{j_n}^{N(t_n)} dN(t_1) \cdots dN(t_n)\Big] \quad (6.50)$$

The expectation entering (6.50) may be recast as

$$E\big[\lambda_{j_1}^{N(t_1)}\lambda_{j_2}^{N(t_1)+N(t_1,t_2)}\cdots\lambda_{j_n}^{N(t_1)+N(t_1,t_2)+\cdots+N(t_{n-1},t_n)}dN(t_1)\cdots dN(t_n)\big] =$$

$$E\big[(\lambda_{j_1}\lambda_{j_2}\cdots\lambda_{j_n})^{N(t_1)}(\lambda_{j_2}\cdots\lambda_{j_n})^{N(t_1,t_2)}\cdots$$

$$(\lambda_{j_{n-1}}\lambda_{j_n})^{N(t_{n-2},t_{n-1})}\lambda_{j_n}^{N(t_{n-1},t_n)}dN(t_1)\cdots dN(t_n)\big] \tag{6.51}$$

where $N(t_r,t_{r+1}) = N(t_{r+1}) - N(t_r)$. Splitting of the expectation, with due account of the overlapping of $n-1$ intervals, may be performed as follows

$$E\Big[(\lambda_{j_{r+1}}\cdots\lambda_{j_n})^{N(t_r,t_{r+1})}dN(t_r)\Big] =$$

$$E\Big[(\lambda_{j_{r+1}}\cdots\lambda_{j_n})^{N(t_r,t_r+dt_r)+N(t_r+dt_r,t_{r+1})}dN(t_r)\Big] =$$

$$E\Big[(\lambda_{j_{r+1}}\cdots\lambda_{j_n})^{N(t_r,t_r+dt_r)}dN(t_r)\Big]E\Big[(\lambda_{j_{r+1}}\cdots\lambda_{j_n})^{N(t_r+dt_r,t_{r+1})}\Big] =$$

$$\lambda_{j_{r+1}}\cdots\lambda_{j_n}\nu E\Big[(\lambda_{j_{r+1}}\cdots\lambda_{j_n})^{N(t_r+dt_r,t_{r+1})}\Big]dt_r \tag{6.52}$$

for $r = 1,\ldots,n-1$. Consequently the expression for the product density becomes

$$f_n(t_1,\ldots,t_n) = \nu^n\sum_{j_1=1}^{k}\cdots\sum_{j_n=1}^{k}b_{j_1}\cdots b_{j_n}\lambda_{j_2}\lambda_{j_3}^2\cdots\lambda_{j_{n-1}}^{n-2}\lambda_{j_n}^{n-1}$$

$$\exp\Big(\nu(t_n-t_{n-1})(\lambda_{j_n}-1)\Big)\exp\Big(\nu(t_{n-1}-t_{n-2})(\lambda_{j_n}\lambda_{j_{n-1}}-1)\Big)\cdots$$

$$\exp\Big(\nu(t_2-t_1)(\lambda_{j_2}\cdots\lambda_{j_n}-1)\Big)\exp\Big(\nu t_1(\lambda_{j_1}\lambda_{j_2}\cdots\lambda_{j_n}-1)\Big) \tag{6.53}$$

6.2.2 Modelling technique for renewal processes

Assume that the interarrival time I is formed as a sum of k independent random variables E_j, i.e.

$$I = \sum_{j=1}^{k}E_j \tag{6.54}$$

where $E_j \sim E(\nu_j)$ is assumed to be an exponentially distributed random variable with parameter ν_j. Let us call such a renewal process a generalized Erlang process. If $k=1$, then $I \sim E(\nu)$ and a homogeneous Poisson counting process is obtained. If $\nu_1 = \nu_2 = \cdots = \nu_k = \nu$ then $I \sim \mathrm{G}(k-1,\nu)$ and an Erlang renewal process is obtained. The transformation of such a process to an equivalent Poisson process at the expense of introduction of extra auxiliary state variables similar to Y_j has been considered in the

section 6.1. Allowing for different parameters of E_j a much larger class of interarrival time distributions can be modelled by suitable choice of the parameters ν_j. The Laplace transform of the p.d.f. $f_I(t)$ of the interarrival time I is

$$f_I(s) = \prod_{j=1}^{k} f_{I_j}(s) = \prod_{j=1}^{k} \frac{\nu_j}{s+\nu_j} \tag{6.55}$$

The renewal density then becomes, cf. (4.41)

$$\begin{aligned} h_o(t) &= \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} \frac{f_I(s)}{1-f_I(s)} \mathrm{e}^{st} ds = \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} \frac{\prod_{j=1}^{k} \nu_j}{\prod_{j=1}^{k}(s+\nu_j) - \prod_{j=1}^{k} \nu_j} \mathrm{e}^{st} ds \\ &= \prod_{j=1}^{k} \nu_j \cdot \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} \frac{\mathrm{e}^{st}}{\prod_{j=0}^{k-1}(s-s_j)} ds = \prod_{j=1}^{k} \nu_j \cdot \sum_{l=0}^{k-1} \frac{\mathrm{e}^{s_l t}}{\prod_{\substack{j=0 \\ j\neq l}}^{k-1}(s_l - s_j)} \end{aligned} \tag{6.56}$$

where s_j, $j = 0, \ldots, k-1$ are the roots of the denominator polynomial

$$\prod_{j=1}^{k}(s+\nu_j) - \prod_{j=1}^{k} \nu_j = 0 \tag{6.57}$$

and β is chosen arbitrarily, so $\beta > \mathrm{Re}(s_j)$, $j = 0, \ldots, k-1$. Notice that $s_0 = 0$ is always a root of (6.57). The last integral of (6.56) can be evaluated by the method of residues. In the same way the pdf of the interarrival time of impulses can be obtained via inverse Laplace transformation of (6.55)

$$f_I(t) = \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} \frac{\prod_{j=1}^{k} \nu_j}{\prod_{j=1}^{k}(s+\nu_j)} \mathrm{e}^{st} ds = \prod_{j=1}^{k} \nu_j \cdot \sum_{l=1}^{k} \frac{\mathrm{e}^{-\nu_l t}}{\prod_{\substack{j=1 \\ j\neq l}}^{k}(\nu_j - \nu_l)} \tag{6.58}$$

The parameters $\nu_1, \ldots, \nu_k$ should be chosen, so (6.58) fits a given target distribution, which will be illustrated in the example problem. Next, the conditions are investigated under which the present renewal process can be represented by a compound Poisson driven process as specified by eqs. (6.44). The first step is to fit the product densities of 1st degree following from (6.53)

$$f_1(t) = \nu \sum_{l=0}^{k-1} b_l \mathrm{e}^{\nu t(\lambda_l - 1)} \tag{6.59}$$

According to (6.40), (6.56) and (6.59) should be equal, which provides the following solution for b_l and λ_l, $l = 0, \ldots, k-1$

$$b_l = \frac{1}{\nu} \frac{\prod_{j=1}^{k} \nu_j}{\prod_{\substack{j=0 \\ j \neq l}}^{k-1} (s_l - s_j)} = \frac{1}{\nu} \frac{1}{\sum_{j=1}^{k} \frac{1}{s_l + \nu_j}} \tag{6.60}$$

$$\lambda_l = 1 + \frac{s_l}{\nu} \tag{6.61}$$

In the last statement of (6.60) the following result has been utilized, cf. (6.57)

$$\prod_{\substack{j=0 \\ j \neq l}}^{k-1} (s_l - s_j) = \lim_{s \to s_l} \frac{\prod_{j=1}^{k} (s + \nu_j) - \prod_{j=1}^{k} \nu_j}{s - s_l} \lim_{s \to s_l} \frac{d}{ds} \prod_{j=1}^{k} (s + \nu_j) = \sum_{j=1}^{k} \frac{\prod_{m=1}^{k} \nu_m}{s_l + \nu_j} \tag{6.62}$$

Setting $N(0) = 0$ the initial values of (6.44) follow from (6.41), (6.46), (6.60) and (6.61) as

$$Y_l(0) = \sum_{j=0}^{k-1} b_j \lambda_j^{l-1} = \frac{1}{\nu} \sum_{j=0}^{k-1} \frac{\left(1 + \frac{s_j}{\nu}\right)^{l-1}}{\sum_{m=1}^{k} \frac{1}{s_j + \nu_m}} \quad , \quad l = 1, 2, \ldots, k-1 \tag{6.63}$$

With known λ_l the final step is to determine the coefficients $\frac{a_0}{a_k}, \frac{a_1}{a_k}, \ldots, \frac{a_{k-1}}{a_k}$ entering the stochastic differential equations (6.44). The latter of these fractions form the invariants of (6.48), which may be obtained from the expansion

$$\prod_{j=1}^{k-1} (\lambda - \lambda_j) = \lambda^{k-1} + \frac{a_{k-1}}{a_k} \lambda^{k-2} + \cdots + \frac{a_1}{a_k} \tag{6.64}$$

where λ_j are given by (6.61). This requires the solution of (6.57). However, there is no need to solve (6.57) for getting the coefficients $\frac{a_j}{a_k}$, $j = 1, \ldots, k-1$. Actually, upon insertion of (6.61) in the left-hand side of (6.63) one has from (6.57)

$$\prod_{j=1}^{k-1} \left(\lambda - 1 - \frac{s_j}{\nu}\right) = \frac{1}{\nu^k (\lambda - 1)} \prod_{j=0}^{k-1} \left(\nu(\lambda - 1) - s_j\right) =$$

$$\frac{1}{\nu^k (\lambda - 1)} \left(\prod_{j=1}^{k} \left(\nu(\lambda - 1) + \nu_j\right) - \prod_{j=1}^{k} \nu_j \right) \tag{6.65}$$

If the right-hand side of (6.65) is expanded and compared with the right-hand side of (6.64), a direct solution for $\frac{a_j}{a_k}$, $j = 1, \ldots, k-1$ is obtained. $\frac{a_0}{a_k}$ can finally be calculated from (6.47), (6.60).

The state vector augmented by the auxiliary variables is governed by the stochastic differential equations

$$d\mathbf{Z}(t) = \mathbf{c}\big(\mathbf{Z}(t), t\big)dt + \mathbf{b}\big(\mathbf{Z}(t), t, P(t)\big)dN(t) \tag{6.66}$$

$$\left.\begin{aligned} \mathbf{Z}(t) &= \begin{bmatrix} \mathbf{Z}_1(t) \\ \mathbf{Z}_2(t) \end{bmatrix}, \quad \mathbf{c}\big(\mathbf{Z}(t), t\big) = \begin{bmatrix} \mathbf{c}_1\big(\mathbf{Z}_1(t), t\big) \\ \mathbf{0} \end{bmatrix} \\ \mathbf{b}\big(\mathbf{Z}(t), t, P(t)\big) &= \begin{bmatrix} \mathbf{b}_1\big(\mathbf{Z}_1(t), t\big) Y_1(t) P(t) \\ \mathbf{b}_2\big(\mathbf{Z}_2(t), t\big) \end{bmatrix} \end{aligned}\right\} \tag{6.67}$$

$$\mathbf{Z}_2(t) = \begin{bmatrix} Y_1(t) \\ \vdots \\ Y_{k-2}(t) \\ Y_{k-1}(t) \end{bmatrix}, \mathbf{b}_2(\mathbf{Z}_2(t)) = \begin{bmatrix} Y_2(t) - Y_1(t) \\ \vdots \\ Y_{k-1}(t) - Y_{k-2}(t) \\ -\Big(\frac{a_0}{a_k} + \frac{a_1}{a_k} Y_1(t) + \cdots + \frac{a_{k-2}}{a_k} Y_{k-2}(t) + \big(1 + \frac{a_{k-1}}{a_k}\big) Y_{k-1}(t)\Big) \end{bmatrix} \tag{6.68}$$

$P(t)$ assumes the values $P(t_i) = P_i$ at the times t_i of the Poisson events and P_i are mutually independent and identically distributed as P. The equations for the mean values and joint centralized moments of 2nd, 3rd and 4th order are written as (5.31) – (5.33).

If the drift vector $\mathbf{c}_1\big(\mathbf{Z}_1(t), t\big)$ and diffusion vector $\mathbf{b}_1\big(\mathbf{Z}_1(t), t\big)$ are polynomial nonlinear functions of the structural state vector $\mathbf{Z}_1(t)$, joint central moments of higher order than the provided moment equations appear at the right-hand side of these equations. Then a cumulant neglect closure at the order $N = 4$ will be used. In case of dense pulse arrivals an ordinary cumulant neglect closure scheme may be applied, whereas in case of sparse pulse arrivals a modified scheme may improve the stability and accuracy during the transient initial phase, [6.5].

Example 6.2: Response of a Duffing oscillator to a renewal driven impulse process excitation

A Duffing oscillator subjected to a compound renewal process is considered. Then

$$\mathbf{Z}_1(t) = \begin{bmatrix} Y(t) \\ \dot{Y}(t) \end{bmatrix}, \mathbf{c}_1(\mathbf{Z}_1(t)) = \begin{bmatrix} \dot{Y}(t) \\ -2\zeta\omega_0 \dot{Y}(t) - \omega_0^2 (Y(t) + \varepsilon Y^3(t)) \end{bmatrix}, \mathbf{b}_1(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{6.69}$$

where $Y(t)$ and $\dot{Y}(t)$ denote the displacement and velocity response of the oscillator, ω_0 and ζ are the circular eigenfrequency and damping ratio of the corresponding linear oscillator, and ε is the nonlinearity parameter. The actual renewal process will be replaced by a generalized Erlang process of the order $k = 2$. Then the pdf mean value μ_I and variational coefficient V_I of I become, cf. (6.58)

$$f_I(t) = \frac{\nu_1 \nu_2}{\nu_2 - \nu_1} \left(\mathrm{e}^{-\nu_1 t} - \mathrm{e}^{-\nu_2 t}\right) \tag{6.70}$$

$$\mu_I = E[I] = \frac{\nu_1 + \nu_2}{\nu_1 \nu_2} \ , \ V_I = \frac{\sigma_I}{\mu_I} = \frac{\sqrt{\nu_1^2 + \nu_2^2}}{\nu_1 + \nu_2} \tag{6.71}$$

Assume that the actual renewal process has lognormally distributed interarrival times, so $\nu I \sim LN(\mu, \sigma^2)$, where μ and σ are non-dimensional parameters. The mean value and variational coefficient of I then become

$$\mu_I = \frac{1}{\nu} \exp\left(\mu + \frac{1}{2}\sigma^2\right) \ , \ V_I = \sqrt{\exp(\sigma^2) - 1} \tag{6.72}$$

With μ_I and V_I of the actual distribution given, ν_1 and ν_2 can be determined from (6.70) and (6.71). Assume $\nu\mu_I = \frac{4}{3}$, $V_I = \sqrt{\frac{5}{8}} \Rightarrow \nu_1 = 0.5\nu$, $\nu_2 = 1.5\nu$ and $\mu = \ln\sqrt{\frac{512}{117}}$, $\sigma = \sqrt{\ln\frac{13}{8}}$. In Fig. 6.4 the actual lognormal pdf for I is denoted by -.-.-.-, and the approximating generalized Erlang pdf resulting from this calibration procedure – by —. The data used in the example are $\omega_0 = 1s^{-1}$, $\zeta = 0.05$, $\varepsilon = 0.5$. The renewal density of the generalized Erlang process, with $k = 2$ is obtained as

$$h_o(t) = \frac{\nu_1 \nu_2}{\nu_1 + \nu_2}\left(1 - \exp\left(-(\nu_1 + \nu_2)t\right)\right) \tag{6.73}$$

hence the asymptotic mean arrival rate of impulses is $\frac{\nu_1\nu_2}{\nu_1+\nu_2}$. This quantity is chosen as $10\omega_0$, corresponding to an average number 20π of impulses of the generalized Erlang process per linear eigenperiod T_0, which means a very dense impulse train. The strengths of the impulses are assumed to be Rayleigh distributed, with parameter σ_P chosen in such a way that the original lognormal distributed and generalized Erlang renewal impulse processes have both the same mean square excitation level as the Poisson impulse process with the same mean arrival rate. Hence it is required that the response variance of the corresponding linear oscillator subjected to a Poisson impulse process with the mean arrival rate $\frac{\nu_1\nu_2}{\nu_1+\nu_2}$ should have the value 1, cf. [6.1], [6.5], $\frac{\nu_1\nu_2}{\nu_1+\nu_2}\frac{E[P^2]}{4\zeta\omega_0^3} = \frac{\nu_1\nu_2}{\nu_1+\nu_2}\frac{\sigma_P^2}{2\zeta\omega_0^3} = 1 \Rightarrow \sigma_P = 0.1$. In the analytical technique the parameters b_l and λ_l of the Poisson driven process are assumed in such a way that first degree product density $f_1(t)$ as given by (6.59) equals $h_o(t)$ given by (6.73). This allows the evaluation of the coeffficients $\frac{a_0}{a_2}$ and $\frac{a_1}{a_2}$ of the stochastic equation for the auxiliary state variable. Next the mean arrival rate ν of the underlying Poisson process is assumed as $\nu = (\nu_1 + \nu_2)/2 = \frac{80}{3}\omega_0$. Then it appears that the second and higher degrees product densities as given by (6.53) split to the product form (6.40). Hence, the Poisson driven process becomes a renewal process with a renewal density

$$h_o(t) = \frac{\nu_1 \nu_2}{2\nu}\left(1 - \exp(-2\nu t)\right) \tag{6.74}$$

hence the obtained process is a kind of an Erlang impulse process with strengths reduced by $\frac{\nu_1\nu_2}{\nu^2}$. An Erlang impulse process with the same mean square excitation level should have the impulses strengths evaluated from the condition $\frac{\nu}{2}\frac{\sigma_P^2}{2\zeta\omega_0^3} = 1 \Rightarrow \sigma_P = \sqrt{3}/20$. However, since the strengths of the obtained process are reduced with respect to Erlang process, the assumed value of σ_P must be multiplied by $\frac{\nu^2}{\nu_1\nu_2}$ which yields $\sigma_P = \sqrt{3}/15 = 0.1154667$.

Since here the centralized drift vector components are cubic and the diffusion vector components are linear forms in the state variables, 5th and 6th order moments appear in the equations for joint central moments of order up to fourth. Because of the high mean arrival rate $\nu = \frac{80}{3}\omega_0$ of impulses, these moments have been evaluated by means of an ordinary cumulant neglect closure scheme, [6.1].

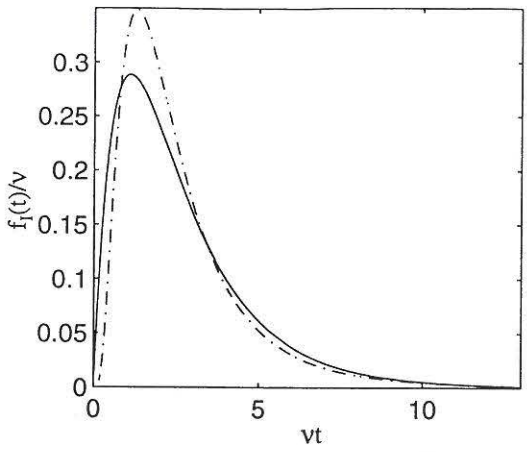

Fig. 6.4. Pdf of interarrival time of renewal processes with $\nu\mu_I = \frac{4}{3}$, $V_I = \sqrt{\frac{5}{8}}$.

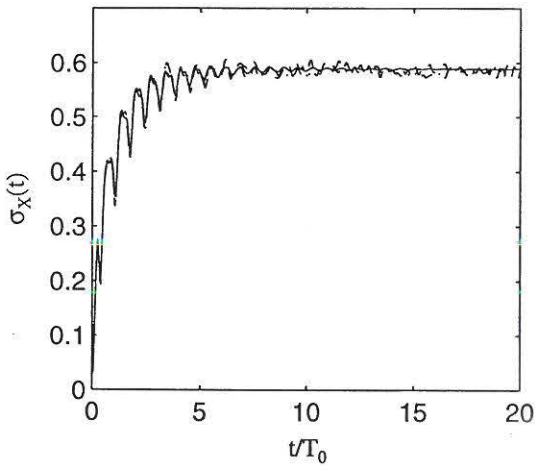

Fig. 6.5. Mean value $\mu_X(t)$ of displace-. ment.

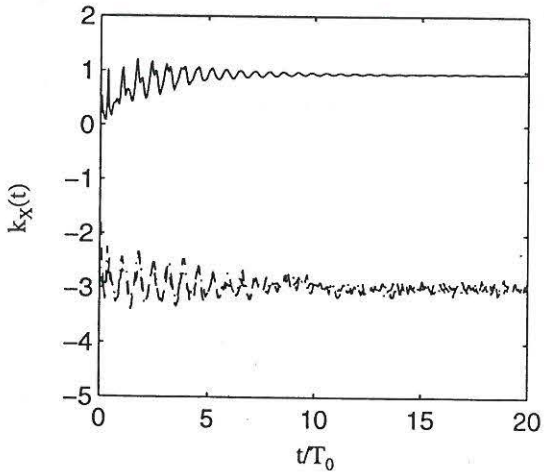

Fig. 6.6. Standard deviation $\sigma_X(t)$ of displacement.

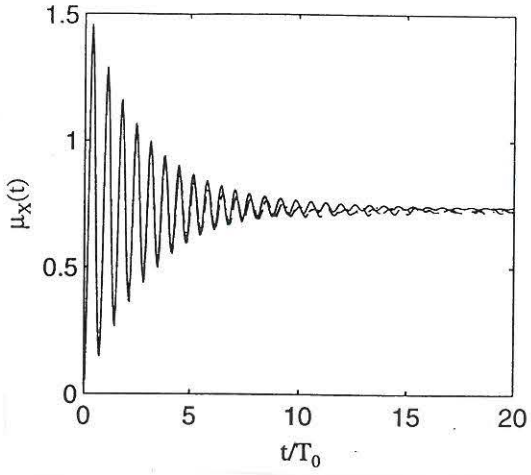

Fig. 6.7. Skewness $S_X(t)$ of displacement.

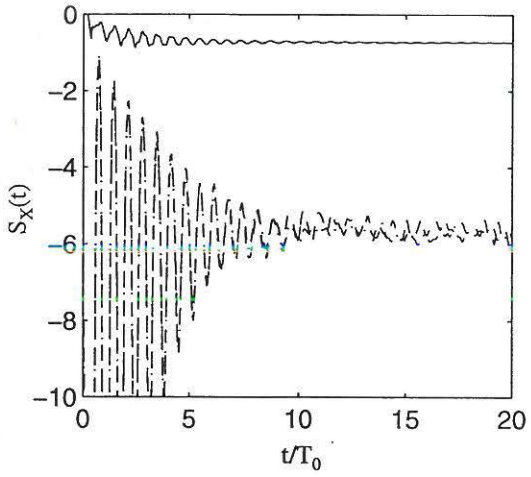

Fig. 6.8. Kurtosis $k_X(t)$ of displacement.

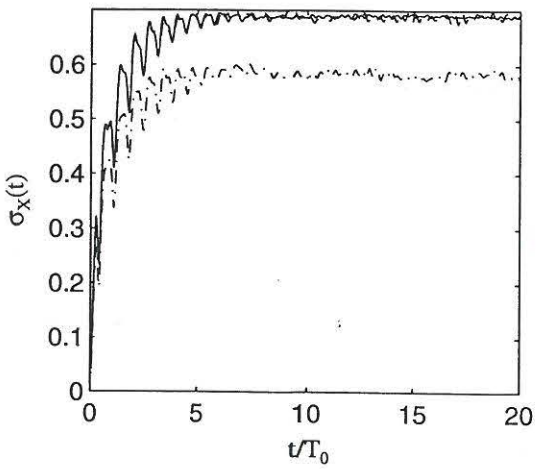

Fig. 6.9. Standard deviation $\sigma_X(t)$ of displacement, for various equivalent excitations.

In order to evaluate the level of approximation introduced both at the replacement of the actual renewal process with a generalized Erlang process and at the cumulant neglect closure procedure, Monte Carlo simulations have been performed, both for lognormally and Erlang distributed interarrival times, [6.7]. The simulated results are obtained based on averaging over 500000 independent response curves, each obtained by numerical integration of the governing equations of motion (6.66) – (6.69).

The results for the mean value $\mu_X(t) = \mu_1(t)$, the standard deviation $\sigma_X(t) = \sqrt{\kappa_{11}(t)}$, the skewness $S_X(t) = \frac{\kappa_{111}(t)}{\sigma_X^3(t)}$ and the kurtosis $k_X(t) = \frac{\kappa_{1111}(t)}{\sigma_X^4(t)} - 3$ are shown in Figs. 6.5 –6.8 as a function of the non-dimensional time $\frac{t}{T_0}$. In the Figures -.-.-.- and - - - - denote Monte Carlo simulation results for lognormally distributed interarrival times, for the generalized Erlang process of order $k = 2$, respectively, and — represents the analytical results for the Poisson driven process obtained for a generalized Erlang process. As seen the agreement between the Monte Carlo simulation results obtained for both renewal processes is very good. Although only the mean value and variance of these distributions are alike, identical response moments within acceptable accuracy are obtained up to and including order 4. Obviously even better results can be obtained if a generalized Erlang process of higher order than $k = 2$ is applied and suitably calibrated. The agreement between the analytical and the Monte Carlo results is very good in the case of the mean values and standard deviations, but it is less satisfactory in the case of skewness and kurtosis coefficients (higher order moments), which can be attributed to the closure of the hierarchy of moment equations at the order $N = 4$. More accurate analytical solutions for these quantities require closure at higher level.

In Fig. 6.9 is shown the displacement standard deviation obtained from the Monte Carlo simulation in cases where the system is subjected to a non-zero mean Gaussian white noise excitation $F(t) = \frac{\nu_1\nu_1}{\nu_1+\nu_2}E[P] + \sqrt{\frac{\nu_1\nu_1}{\nu_1+\nu_2}E[P^2]}W(t)$ (unbroken line —), to a compound Poisson process with the mean arrival rate $\frac{\nu_1\nu_1}{\nu_1+\nu_2}$ (- - - -) and to a renewal process with the lognormally distributed interarrival times (-.-.-.-.). A zero-mean, unit intensity Gaussian white noise process $W(t)$, was generated by the method of Penzien, [6.8]. In all cases 500000 sample curves are used. Because of the high mean arrival rate of impulses, the non-zero mean white noise process and the compound Poisson process produce, as expected, almost identical results. In contrast, the standard deviation for the renewal process is significantly different. In combination to the results obtained in Figs. 6.5–6.8 it is then concluded that the distribution of the interarrival time affects the response moments significantly, but primarily through its mean value and the variance, whereas higher order moments of the interarrival time distribution seem to have less influence.

6.3 Markov approach in the case of Poisson train of general pulses

6.3.1 Statement of the problem: stochastic equations

Response statistical moments of a linear dynamical system subjected to a train of general pulses can be, in principle, exactly evaluated. As the linear superposition principle holds, the cumulative response to a train of pulses is just the effect of superimposing the responses to individual pulses of excitation. In other words, the response is a filtered point process obtained as the effect of filtering the pulse train od excitation through the system. To evaluate the moments it suffices, as it is shown in the section 4.2 to perform the averaging of pertinent multi-fold integrals with respect to the underlying stochastic point (random counting) process. Such a procedure, although rather straightforward, is cumbersome, especially for higher order moments. It is therefore appealing to use an alternative technique of obtaining the moments directly as the solution of pertinent

differential equations. Non-diffusive Markov process technique, in particular, the technique of equations for moments, can be extended to the problem of general pulse if the train of general pulses can be represented as the response of an auxiliary linear system to the train of Dirac delta impulses. This converts the non-Markov problem of the original system driven by the train of general pulses to the Markov problem of an augmented system driven by the train of impulses. Approach of this kind was used in [6.5], where the train of general pulses was idealized as the response of an auxiliary, highly damped, linear filter to the train of impulses. The pulse obtained by such a filtering, although semi-infinite, was due to the high damping a single half-wave with practically insignificant, fast decaying tail.

The following approach due to Ricciardi [6.9] shows that a general pulse, with a single- or multi- sine-half-wave shape, can be exactly represented as the response of an auxiliary linear filter to two superimposed trains of Dirac delta impulses. This approach was developed in connection with analysis of bridges behaviour under random trains of moving loads.

Consider a non-linear (Duffing) oscillator under external excitation in form of a Poisson driven train of general pulses. The state vector of the system $\mathbf{Z}_1(t)$ is governed by the set of equations

$$\frac{d}{dt}\mathbf{Z}_1(t) = \mathbf{c}_1\big(\mathbf{Z}_1(t), t\big) + \mathbf{b}_1 \sum_{i=1}^{N(t)} P_i w(t - t_i) \tag{6.75}$$

The specific, single sine-half-wave shape of duration T_d can be represented as the superposition

$$w(t - t_i) = h(t - t_i) + h(t - t_i - T_d) \tag{6.76}$$

where $h(t)$ is the impulse response function of an undamped linear oscillator, with a natural period $T_f = 2T_d$, hence with an undamped natural frequency $\omega_f = \pi/T_d$. It is easy to note that due to the time shift by a half-period $T_f/2 = T_d$, the undamped response is truncated after that time. The train of general pulses is then represented as

$$\sum_{i=1}^{N(t)} P_i w(t - t_i) = \sum_{i=1}^{N(t)} P_i h(t - t_i) + \sum_{i=1}^{N(t)} P_i h(t - t_i - T_d) \tag{6.77}$$

The stochastic equations, for $t < T_d$, are

$$d\mathbf{Z}(t) = \mathbf{c}\big(\mathbf{Z}(t)\big)\,dt + \mathbf{b}P(t)dN(t) \tag{6.78}$$

and for $t > T_d$ they become

$$d\mathbf{Z}(t) = \mathbf{c}\big(\mathbf{Z}(t)\big)\,dt + \mathbf{b}P(t)dN(t) + \mathbf{b}P(t - T_d)dN(t - T_d) \tag{6.79}$$

where

$$\mathbf{Z}(t) = \begin{bmatrix} Y(t) \\ \dot{Y}(t) \\ X(t) \\ \dot{X}(t) \end{bmatrix}, \mathbf{c}(\mathbf{Z}(t)) = \begin{bmatrix} \dot{Y}(t) \\ -2\zeta\omega_0\dot{Y}(t) - \omega_0^2\big(Y(t) + \varepsilon Y^3(t) + X(t)\big) \\ \dot{X}(t) \\ -\omega_f^2 X(t) \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (6.80)$$

In the present case the differential rule takes the form, cf. (2.89)

$$df(\mathbf{Z}(t)) = \sum_{i=1}^{n} \frac{\partial f(\mathbf{Z}(t))}{\partial Z_i} c_i(\mathbf{Z}(t))dt + \Big(f\big(\mathbf{Z}(t) + \mathbf{b}P(t)\big) - f\big(\mathbf{Z}(t)\big)\Big)dN(t) +$$

$$\Big(f\big(\mathbf{Z}(t) + \mathbf{b}P(t - T_d)\big) - f\big(\mathbf{Z}(t)\big)\Big)dN(t - T_d) \quad (6.81)$$

6.3.2 Differential equations for moments for linear systems

Equations for the mean values, for $t > T_d$, are

$$\dot{\boldsymbol{\mu}}(t) = E\big[\mathbf{c}\big(\mathbf{Z}(t)\big)\big] + \mathbf{b}\big(\nu(t)E[P(t)] + \nu(t - T_d)E[P(t - T_d)]\big) \quad (6.82)$$

and if the impulse process is stationary

$$\dot{\boldsymbol{\mu}}(t) = E\big[\mathbf{c}\big(\mathbf{Z}(t)\big)\big] + 2\mathbf{b}\nu E[P] \quad (6.83)$$

For the second-order moments $E[Z_i(t)Z_j(t)] = \mu_{ij}(t)$ the differential rule (6.81) yields

$$d\mu_{ij}(t) = 2\big\{E\big[Z_i(t)\big(c_j(\mathbf{Z}(t))\big]\big\}_s dt +$$

$$2\big\{\mu_i(t)b_j\big\}_s \nu(t)E[P(t)]dt + 2\big\{E\big[Z_i(t)P(t - T_d)dN(t - T_d)\big]b_j\big\}_s +$$

$$b_i b_j\Big(\nu(t)E\big[P^2(t)\big] + \nu(t - T_d)E\big[P^2(t - T_d)\big]\Big)dt \quad (6.84)$$

The straightforward splitting of the expectation $E[Z_i(t)P(t - T_d)dN(t - T_d)]$ is not feasible due to the overlapping of the random quantities in the interval $\tau \in [t - T_d, t - T_d + d\tau[$. In order to perform the splitting correctly let us express the state variable $Z_i(t)$ (the response of a linear system) in terms of the impulse response function $h_i(t)$ as

$$Z_i(t) = b_i \int_0^t h_i(t - \tau)P(\tau)dN(\tau) + b_i \int_{T_d}^t h_i(t - \tau)P(\tau - T_d)dN(\tau - T_d) \quad (6.85)$$

hence the required expectation is

$$E\big[Z_i(t)P(t-T_d)dN(t-T_d)\big] = b_i \int_0^t h_i(t-\tau)E\big[P(\tau)dN(\tau)P(t-T_d)dN(t-T_d)\big]+$$

$$b_i \int_{T_d}^t h_i(t-\tau)E\big[P(\tau-T_d)dN(\tau-T_d)P(t-T_d)dN(t-T_d)\big] \tag{6.86}$$

The expectations in the integrands can be split except for the interval $\tau \in [t-T_d, t-T_d+d\tau[$, hence

$$E\big[Z_i(t)P(t-T_d)dN(t-T_d)\big] = b_i \int_0^{t-T_d} h_i(t-\tau)E\big[P(\tau)dN(\tau)\big]E\big[P(t-T_d)dN(t-T_d)\big]+$$

$$b_i h_i\big(t-(t-T_d)\big)E\big[P^2(t-T_d)\big(dN(t-T_d)\big)^2\big]+$$

$$b_i \int_{t-T_d+d\tau}^t h_i(t-\tau)E\big[P(\tau)dN(\tau)\big]E\big[P(t-T_d)dN(t-T_d)\big]+$$

$$b_i \int_{T_d}^t h_i(t-\tau)E\big[P(\tau-T_d)dN(\tau-T_d)\big]E\big[P(t-T_d)dN(t-T_d)\big] \tag{6.87}$$

The second, non-integral terms at the right side of (6.87) appears due to to taking into consideration the overlapping interval. As this term has been excluded from the integral let us include again the missing term $b_i h_i\big(t-(t-T_d)\big)E\big[P(t-T_d)\big]\nu(t-T_d)d\tau E\big[P(t-T_d)\big]\nu(t-T_d)dt$, which is of order $\mathrm{O}(dt^2)$.

The required expectation then becomes

$$E\big[Z_i(t)P(t-T_d)dN(t-T_d)\big] = b_i \int_0^{t-T_d} h_i(t-\tau)E\big[P(\tau)\big]\nu(\tau)d\tau E\big[P(t-T_d)\big]\nu(t-T_d)dt +$$

$$b_i h_i\big(T_d\big)E\big[P^2(t-T_d)\big]\nu(t-T_d)dt+$$

$$b_i h_i\big(t-(t-T_d)\big)E\big[P(t-T_d)\big]\nu(t-T_d)d\tau E\big[P(t-T_d)\big]\nu(t-T_d)dt +$$

$$b_i \int_{t-T_d+d\tau}^t h_i(t-\tau)E\big[P(\tau)\big]\nu(\tau)d\tau E\big[P(t-T_d)\big]\nu(t-T_d)dt +$$

$$b_i \int_{T_d}^t h_i(t-\tau)E\big[P(\tau-T_d)\big]\nu(\tau-T_d)d\tau E\big[P(t-T_d)\big]\nu(t-T_d)dt \tag{6.88}$$

The result is

$$E\big[Z_i(t)P(t-T_d)dN(t-T_d)\big] =$$

$$E\big[Z_i(t)\big]E\big[P(t-T_d)\big]\nu(t-T_d)dt + b_i h_i(T_d)E\big[P^2(t-T_d)\big]\nu(t-T_d)dt \tag{6.89}$$

and if the Poisson process is stationary

$$E\big[Z_i(t)P(t-T_d)dN(t-T_d)\big] = E\big[Z_i(t)\big]\nu E[P]dt + b_i h_i(T_d)\nu E\big[P^2\big]dt \tag{6.90}$$

Consequently, in the case of the stationary Poisson process, the equations for second order moments are obtained as

$$\dot{\mu}_{ij}(t) = 2\big\{E\big[Z_i(t)c_j(\mathbf{Z}(t))\big]\big\}_s +$$

$$4\big\{\mu_i(t)b_j\big\}_s\nu E[P] + 2\nu E\big[P^2\big]\big(b_i b_j + \big\{b_i h_i(T_d)d_j\big\}_s\big) \tag{6.91}$$

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6.5 Example problems

6.1 Derive equations for first and second order moments of the response of a linear oscillator to an Erlang, with $k = 2$, impulse process excitation.

6.2 Formulate the stochastic equations for the centralized state variables corresponding to (6.79) and the equations for second-order central moments corresponding to (6.91).

CHAPTER 7

FIRST–PASSAGE TIME PROBLEMS

7.1 Statement of problems

In fig. 7.1 some realizations of the scalar process $\{Z(t), t \in [t_0, \infty[\}$ are shown, as well as a deterministic upper *limit state function* $b(t)$ which separates unsafe and safe states of the process. The *safe domain* at the time t is given as

$$S_t = \{z \mid -\infty < z < b(t)\} \tag{7.1}$$

Consequently, the limit state function $b(t)$ is considered as part of the *unsafe domain*, S_t^c, which is the complement of S_t. Now, consider a sufficiently large number M of realizations. Of these realizations, M_0 realizations will be in the safe domain S_{t_0} at the time $t = t_0$, and the remainder $M - M_0$ realizations originate from the unsafe domain, see fig. 7.1. During the interval $]t, t + \Delta t]$ a certain number ΔM of realizations are leaving the safe domain. Among these, ΔM_0 realizations originated from the safe domain at the time $t = t_0$, whereas the remaining $\Delta M - \Delta M_0$ ones originated from the unsafe domain. A further distinction will be made among the ΔM_0 realizations. Let ΔM_1 of these cross out for the first time in the interval $]t, t+\Delta t]$, and the remaining $\Delta M_0 - \Delta M_1$ realizations have at least 1 previous out-crossing in the interval $]t_0, t]$. ΔM_1-realizations are called *first-passages* in the interval $]t, t + \Delta t]$. Obviously, $\Delta M_1 \leq \Delta M_0 \leq \Delta M$.

The probability of an out-crossing in $]t, t + \Delta t]$ can now be estimated as the fraction $\frac{\Delta M}{M}$. It is assumed that this fraction becomes proportional to Δt, as $\Delta t \to 0$, i.e.

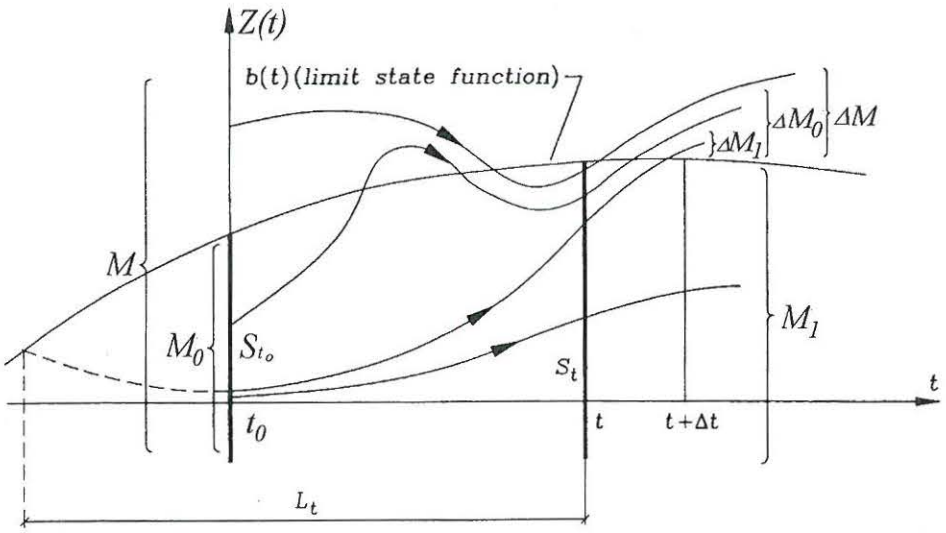

Fig. 7.1. First passages and out-crossings at the time t.

$$\frac{\Delta M}{M} = f_1(t)\Delta t + \mathrm{O}(\Delta t^2) \tag{7.2}$$

where $f_1(t)$ is termed as the 1st order outcrossing rate.

It is also assumed that the probability of two or more out-crossings in $]t, t+\Delta t]$ is negligible compared to the probability of exactly one out-crossing, if Δt is sufficiently small. These probabilities are then at most $\mathrm{O}(\Delta t^2)$. Then, $f_1(t)$ can also be interpreted as the expected number of out-crossings per unit time. In the same way, the fraction $\frac{\Delta M_1}{M_0}$ is assumed to be proportional to Δt, i.e.

$$\frac{\Delta M_1}{M_0} = f_T(t)\Delta t + \mathrm{O}(\Delta t^2) \tag{7.3}$$

where $f_T(t)$ is termed as the *first-passage probability time density function.*

The integral $\int_{t_0}^{t} f_T(\tau)d\tau$ signifies the relative number of realizations originating from the safe domain S_{t_0} at the time t_0, which have left the safe domain during the interval $]t_0, t]$. Obviously this fraction is equal to the probability of failure during the interval $]t_0, t]$ on condition of being in the safe domain S_{t_0}, $P_f(]t_0, t]|Z(t_0) \in S_{t_0})$. Hence,

$$P_f(]t_0, t]|Z(t_0) \in S_{t_0}) = \Pr\{T \in]t_0, t]\} = \int_{t_0}^{t} f_T(\tau)d\tau = F_T(t) \tag{7.4}$$

where $F_T(t)$ is termed as the *first-passage time probability distribution function.*

Since any realization will sooner or later cross out of the safe domain, it follows that $\lim_{t\to\infty} F_T(t) = 1$. The unconditional probability of failure in $]t_0, t]$, $P_f(]t_0, t])$, can next be expressed as

$$\begin{aligned} P_f(]t_0, t]) &= 1 - \Pr\big(Z(t_0) \in S_{t_0}\big) + P_f\big(]t_0, t] \mid Z(t_0) \in S_{t_0}\big) \Pr\big(Z(t_0) \in S_{t_0}\big) = \\ &1 - \Pr\big(Z(t_0) \in S_{t_0}\big) + F_T(t) \Pr\big(Z(t_0) \in S_{t_0}\big) = \\ &1 - F_{\{Z\}}\big(b(t_0), t_0\big)\big(1 - F_T(t)\big) \end{aligned} \tag{7.5}$$

where

$$\Pr\big(Z(t_0) \in S_{t_0}\big) = \Pr\big(Z(t_0) \le b(t_0)\big) = F_{\{Z\}}\big(b(t_0), t_0\big) \tag{7.6}$$

$1 - \Pr\big(Z(t_0) \in S_{t_0}\big)$ indicates the probability of initial failure, i.e. the realizations are originating from the unsafe domain. $F_{\{Z\}}(z,t)$ is the distribution function of the 1st order of the process $\{Z(t), t \in [t_0, \infty[\}$. Among the M_0 realizations, a certain number

M_1 of them has not left the safe domain in the interval $]t_0, t]$, see fig. 7.1. The *hazard rate* $h(t)$ is then defined from the relation

$$h(t)\Delta t + \mathrm{O}(\Delta t^2) = \frac{\Delta M_1}{M_1} \tag{7.7}$$

The right-hand side of (7.7) signifies the number of first passages in $]t, t + \Delta t]$ relative to the number of samples which have not failed in the preceding interval $]t_0, t]$. Hence, $h(t)\Delta t + \mathrm{O}(\Delta t^2)$ can be interpreted in terms of the following conditional probability

$$h(t)\Delta t + \mathrm{O}(\Delta t^2) = \Pr\Big(1 \text{ out-crossing in }]t, t + \Delta t] \mid \forall \tau \in]t_0, t] : Z(\tau) \in S_\tau\Big) \tag{7.8}$$

From (7.3) and (7.7)

$$h(t)\Delta t + \mathrm{O}(\Delta t^2) = \frac{\Delta M_1}{M_0}\frac{M_0}{M_1} = \frac{f_T(t)\Delta t}{1 - F_T(t)} \Rightarrow$$

$$h(t) = \frac{f_T(t)}{1 - F_T(t)} \tag{7.9}$$

where the result $\frac{M_1}{M_0} = 1 - F_T(t)$ has been applied.

(7.9) has the solution

$$F_T(t) = 1 - \exp\Big(-\int_{t_0}^{t} h(\tau)\, d\tau\Big) \tag{7.10}$$

(7.10) can be proved by insertion into (7.9), and using $f_T(t) = \frac{d}{dt}F_T(t)$. Inserting (7.10) into (7.5) provides the following expression for $P_f(]t_0, t])$ in terms of the hazard rate

$$P_f\big(]t_0, t]\big) = 1 - F_{\{Z\}}\big(b(t_0), t_0\big) \exp\Big(-\int_{t_0}^{t} h(\tau)\, d\tau\Big) \tag{7.11}$$

No exact solutions are available for $h(\tau)$ for even the simplest system of engineering significance. Hence, the subject of reliability theory for dynamically excited structures is to specify suitable approximations for the hazard rate for this class of structures. Before doing this, one further useful identity for the first-passage probability density will be derived. From (7.2) and (7.3) follows

$$f_T(t) = \frac{\Delta M_1}{\Delta t\, M_0} = \frac{\Delta M_1}{\Delta M}\frac{M}{M_0}\frac{\Delta M}{\Delta t\, M} = \frac{\Delta M_1}{\Delta M}\frac{M}{M_0} f_1(t) =$$

$$\frac{f_1(t)}{\Pr\big(Z(t_0) \in S_{t_0}\big)} \; \frac{\Delta M_1}{\Delta M} \tag{7.12}$$

where $\Pr\big(Z(t_0) \in S_{t_0}\big) = \frac{M_0}{M}$ has been introduced. In fig. 7.1 with the dashed line, the ΔM_1-realization has been artificially prolonged backwards until the time previous to $t = t_0$, where it crossed into the safe domain. L_t signifies the time-interval spent in the safe domain before an out-crossing at the time t. Among the total number ΔM of out-crossings in $]t, t + \Delta t]$, ΔM_1-realizations are then distinguished by the property that $L_t > t$. Hence, the fraction $\frac{\Delta M_1}{\Delta M}$ can be interpreted as the probability $\Pr(L_t > t)$ that the time-length L_t spent in the safe domain prior to an out-crossing at the time t is larger than t. (7.12) can then be written

$$f_T(t) = \frac{f_1(t)}{\Pr\big(Z(t_0) \in S_{t_0}\big)} \Pr(L_t > t) = \frac{f_1(t)}{\Pr\big(Z(t_0) \in S_{t_0}\big)} \big(1 - F_{L_t}(t)\big) \tag{7.13}$$

where $F_{L_t}(l)$ is the probability distribution function of L_t. Now, if the process $\{Z(t), t \in R\}$ is a stationary process, and the limit state function $b(t)$ is constant with time, the crossing rate $f_1(t)$ as well as the distribution function $F_{L_t}(l)$ become independent of t. Introducing the designation $F_L(l)$ for the time-invariant distribution function, (7.13) can be simplified to

$$f_T(t) = \frac{f_1}{\Pr\big(Z(t_0) \in S\big)} \big(1 - F_L(t)\big) \tag{7.14}$$

Integrating (7.14) from t_0 to ∞, and using integration by parts provide

$$\begin{aligned} 1 = \int_{t_0}^{\infty} f_T(\tau)\, d\tau = \frac{f_1}{\Pr\big(Z(t_0) \in S\big)} \int_{t_0}^{\infty} \big(1 - F_L(\tau)\big)\, d\tau = \\ \frac{f_1}{\Pr\big(Z(t_0) \in S\big)} \left(\Big[\tau\big(1 - F_L(\tau)\big)\Big]_{t_0}^{\infty} + \int_{t_0}^{\infty} \tau f_L(\tau)\, d\tau \right) = \\ \frac{f_1}{\Pr\big(Z(t_0) \in S\big)} E[L] \end{aligned} \tag{7.15}$$

where $E[L] = \int_{t_0}^{\infty} \tau f_L(\tau)\, d\tau$ is the expected value of L. (7.14) can then be written as

$$f_T(t) = \frac{1}{E[L]} \big(1 - F_L(t)\big) \tag{7.16}$$

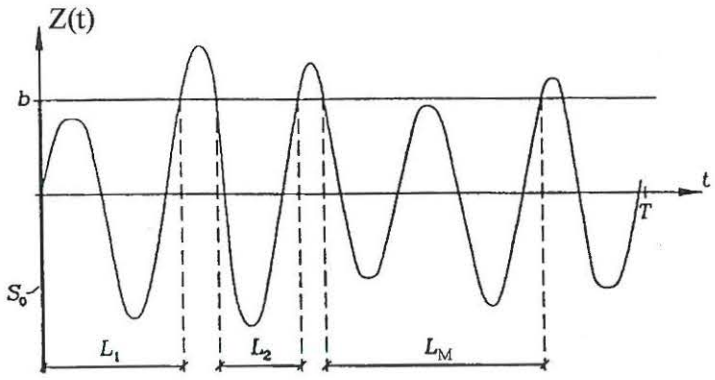

Fig. 7.2. Ergodic sampling of interval lengths spent in the safe domain before an out-crossing.

If the safe domain is time-invariant and $\{Z(t), t \in R\}$ is a stationary process possessing sufficient ergodicity properties, $F_L(l), E[L]$ and $\Pr\big(Z(t_0) \in S\big)$ can all be determined by ergodic sampling from a single realization $z(t)$ of the length T. In what follows it is assumed that $z(t)$ is generated artificially by a computer using the so-called *Monte Carlo simulation*, in a way that $z(t)$ has identical statistical properties to the physical phenomenon, which $\{Z(t), t \in R\}$ is supposed to model. During the interval $[0, T]$, a total of M out-crossings from the safe domain is observed, and the corresponding $L_1, \ldots, L_M$ intervals spent in the safe domain before the out-crossings are measured, see fig. 7.2. Let $M_{\leq l}$ be the number of intervals in the sample thus obtained, for which $L_j \leq l$. The distribution function $F_L(l)$, the expected value $E[L]$ and $\Pr\big(Z(t_0) \in S\big)$ are then estimated from the statistics

$$F_L(l) \simeq \frac{M_{\leq l}}{M} \tag{7.17}$$

$$E[L] \simeq \frac{1}{M} \sum_{j=1}^{M} L_j \tag{7.18}$$

$$\Pr\big(Z(t_0) \in S\big) \simeq \frac{1}{T} \sum_{j=1}^{M} L_j \tag{7.19}$$

If the estimates (7.17) and (7.18) are inserted into (7.16), robust estimates of the first-passage pdf are obtained, because the sampling has been related to a distribution function $F_L(l)$, and not directly to the probability density function $f_T(t)$.

Since $\Delta M_1 \leq \Delta M$ it follows that, cf.(7.12)

$$f_T(t) = \frac{\Delta M_1}{\Delta M} \frac{M}{M_0} f_1(t) \leq \frac{M}{M_0} f_1(t) = \frac{f_1(t)}{\Pr(Z(t_0) \in S_{t_0})} \tag{7.20}$$

It then follows that

$$F_T(t) \leq \frac{1}{\Pr(Z(t_0) \in S_{t_0})} \int_{t_0}^{t} f_1(\tau) d\tau \Rightarrow$$

$$P_f([t_0, t]) = 1 - \Pr(Z(t_0) \in S_{t_0}) + F_T(t) \Pr(Z(t_0) \in S_{t_0}) \leq$$

$$\Pr(Z(t_0) \notin S_{t_0}) + \int_{t_0}^{t} f_1(\tau) d\tau \tag{7.21}$$

Since $\Pr(Z(t_0) \notin S_{t_0})$ and $f_1(\tau)$ in some cases can be calculated, (7.21) provides a useful upper bound for the failure probability.

A distinction will be made between the so-called *stochastic start* and *deterministic start problem.* For a stochastic start a window is opened on the time axis at the arbitrary time t_0 of a system which has been exposed to dynamic loadings prior to the time t_0. For the deterministic start problem the system is at rest at some state $z_0 = z(t_0)$ at the time t_0, which is assumed to be within the safe domainS_{t_0}. Hence for a deterministic start problem $\Pr(Z(t_0) \in S_{t_0}) = 1$ whereas for the stochastic start problem $\Pr(Z(t_0) \in S_{t_0}) < 1$. Obviously the simulation method only applies to the stochastic start problem.

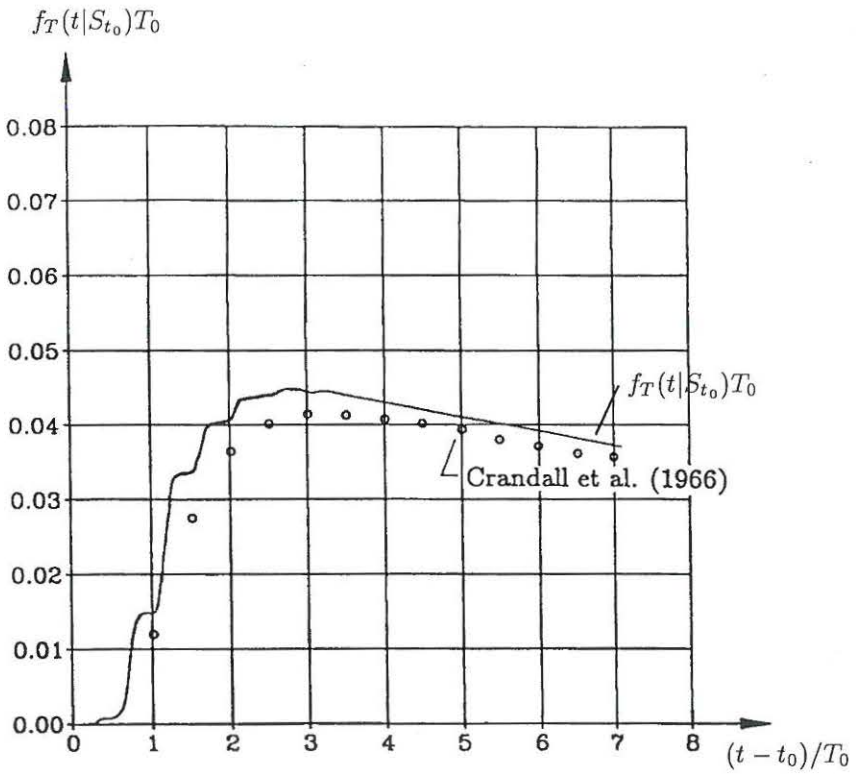

Fig. 7.3. First-passage time probability density function. Deterministic start problem with symmetric constant double barrier. SDOF system, $\zeta = 0.08, b = -a = 2\sigma_{Y,0}, Z(t_0) = \dot{Z}(t_0) = 0$.

The first-passage time problem described in fig. 7.1. is classified as a *single barrier problem.* If also a lower limit state function $a(t)$ is defined one talks about a *double barrier problem.* In this case the safe domain is given by $S_t = \{z|a(t) < z < b(t)\}$ and $\partial S_t = \{z|z = a(t) \vee z = b(t)\}$.

The indicated scalar first-passage time problems can immediately be generalized to the case where the state is defined by the $n-dimensional$ vector process $\{\mathbf{Z}(t), t \in [t_0, \infty[\}$. The safe domain is determined by a subset $S_t \subset R^n$ of the sample space of the state vector $\mathbf{Z}(t)$. The simulation method (7.16) can also be applied in the vector case if only the safe domain is time-invariant and the state vector response is stationary. For both the scalar and vector case, the sample size should be at least $M = 1000$. Hence, the length of the realization of the considered ergodic process should be at least $T = 1000/f_1$.

In fig. 7.3 is shown the first-passage time probability density function for a deterministic start problem with a symmetric constant double barrier for a linear SDOF system subjected to a stationary white noise. $F_T(t)$ increases to a maximum, and eventually decreases to 0, so the area below the curve amounts to 1. The full drawn curve is based on an approximate technique, Nielsen (1980) [7.1]. The simulation results are due to Crandall *et al.* (1966) [7.2].

The corresponding result for a stationary start problem with a constant single barrier problem is shown in fig. 7.4. based on $M = 100000$ out-crossings. In this case $f_T(t)$ is a non-increasing function of time with a characteristic stair-case like behaviour, reflecting the eigenvibration period T_0 of the system.

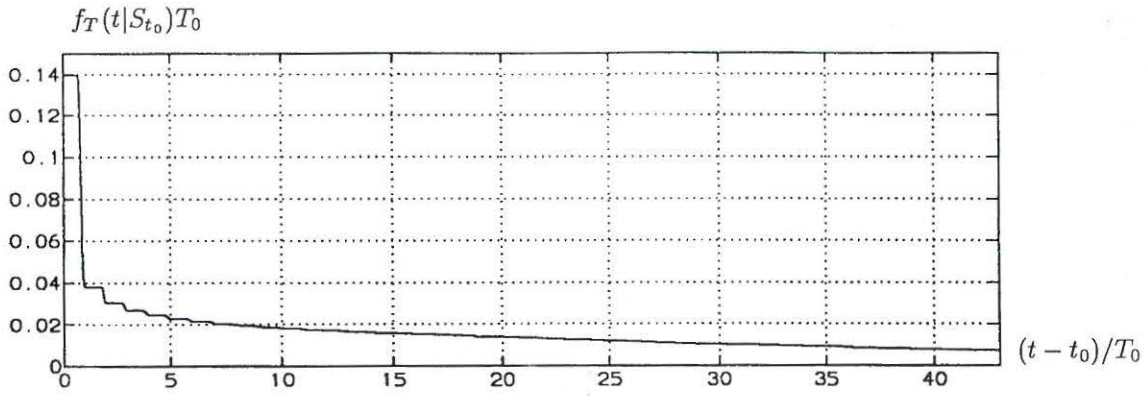

Fig. 7.4. First-passage time probability density function. Stochastic start problem, single constant barrier. SDOF system $\zeta = 0.08, b = -a = 2\sigma_{Y,0}$.

Example 7.1: Extremes of dynamic response processes

Consider a scalar stochastic process $\{Y(\tau), \tau[t_0, t_1]\}$. Of specific importance in structural design are the *maximum value* and the *minimum value* of $\{X(\tau), \tau \in [t_0, t_1]\}$, defined as follows

$$Y_{\max}([t_0, t_1]) = \max_{\tau \in [t_0, t_1]} Y(\tau) \tag{7.22}$$

$$Y_{\min}([t_0, t_1]) = \min_{\tau \in [t_0, t_1]} Y(\tau) \tag{7.23}$$

$Y_{\max}([t_0,t_1])$ and $Y_{\min}([t_0,t_1])$ are called jointly the *extremes* of the process $\{Y(\tau),\tau[t_0,t_1]\}$. The determination of the probability distribution function of the extremes is strongly related to the problem of finding the probability of failure $P_f(]t_0,t_1])$ in the interval $]t_0,t_1]$ with a constant barrier first-passage time problem, as demonstrated below

$Y_{\max}([t_0,t_1])$ is smaller than or equal to some value x, if all stochastic values in the process $\{Y(\tau),\tau \in [t_0,t_1]\}$ fulfil this criterion. Hence, the distribution function of $Y_{\max}([t_0,t_1])$ becomes

$$F_{Y_{\max}}(x;[t_0,t_1]) = \Pr\Big(Y_{\max}([t_0,t_1]) \le x\Big) = \Pr(\forall\tau \in [0,t] : Y(\tau) \le x) \tag{7.24}$$

The last statement of (7.24) expresses the *reliability* of the process $\{Y(\tau),\tau \in [t_0,t_1]\}$ relative to the safe domain $S = \{\xi \mid -\infty < \xi < x\}$, see fig. 7.5 a). This is 1 minus the probability of failure. Then

$$F_{Y_{\max}}(x;[t_0,t_1]) = 1 - P_f(]t_0,t_1];]-\infty,x[) \tag{7.25}$$

where $P_f(t_0,t_1];]-\infty,x[)$ is the probability of failure relative to the considered safe domain. Similarly, $Y_{\min}([t_0,t_1])$ is larger than some fixed value x, if all stochastic variables in $\{Y(\tau),\tau \in [t_0,t_1]\}$ fulfil this criterion. Then

$$\Pr\Big(Y_{\min}([t_0,t_1]) > x\Big) = \Pr(\forall\tau \in [t_0,t_1] : Y(\tau) > x) \tag{7.26}$$

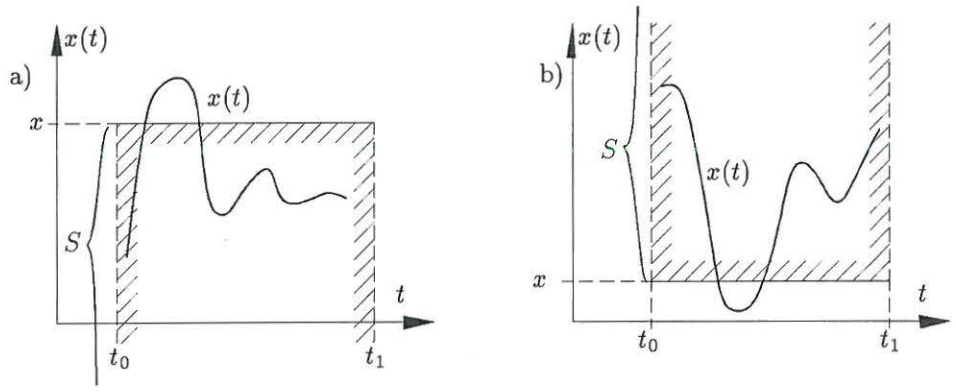

Fig. 7.5. Equivalent safe domain at the specification of the probability distribution function of extreme values. a) Maximum value. b) Minimum value.

The right-hand side represents the reliability of the process $\{Y(\tau),\tau \in [t_0,t_1]\}$ relative to the safe domain $S = \{\xi \mid x < \xi < \infty\}$. Hence

$$F_{Y_{\min}}(x;[t_0,t_1]) = \Pr\left(Y_{\min}([t_0,t_1]) \le x\right) = P_f(]t_0,t_1];]x,\infty[) \tag{7.27}$$

where $P_f(]t_0,t_1];]x,\infty[)$ is the probability of failure of $\{Y(\tau),\tau \in [t_0,t_1]\}$ relative to the considered safe domain. In (7.25) and (7.27), the distribution function of the extremes of a stochastic process has been expressed in terms of the probability of failure of $\{X(\tau),\tau \in [t_0,t_1]\}$ relative to properly defined time-constant single barrier safe domains.

7.2 Markov systems

Assume the system is described by the Markov state vector process $\{\mathbf{Z}(t), t \in [t_0, t_1]\}$ with the transition probability density function $q_{\{\mathbf{Z}\}}(\mathbf{z}, t|\mathbf{x}, t_0), t > t_0$.

Consider a system with the deterministic start in $\mathbf{x} \in S_{t_0}$. Transitions from this state to a later state $\mathbf{z}$ at the time t are governed by the forward integro-differential Chapman-Kolmogorov equation (2.25). The system can only leave the safe domain S_t through the exit part of the boundary $\partial S_t^{(1)}$, and reenter in the safe domain through the entrance part $\partial S_t^{(0)}$. In the reliability problems one is concerned with sample curves which have not left the safe domain in a given interval $]t_0, t]$. Transitions from any point $\mathbf{x} \in S_{t_0}$ in the safe domain at the time t_0 to some point $\mathbf{z} \in \partial S_t^{(0)}$ on the entrance part of the boundary should then be prevented, corresponding to the boundary condition $q_{\{\mathbf{Z}\}}(\mathbf{z}, t|\mathbf{x}, t_0) = 0$. This implies that any realization at the point of reentering the safe domain through the entrance part of the boundary is absorbed or extracted from the sample and all remaining sample curves have never left the safe domain up to the time t. $q_{\{\mathbf{Z}\}}(\mathbf{z}, t|\mathbf{x}, t_0)$ is then seen to fulfill the following boundary and initial value problem, cf. (2.45), (2.47).

$$\left.\begin{aligned} &\frac{\partial}{\partial t} q_{\{\mathbf{Z}\}}(\mathbf{z}, t|\mathbf{x}, t_0) = \mathcal{K}_{\mathbf{z},t}\left[q_{\{\mathbf{Z}\}}(\mathbf{z}, t \mid \mathbf{x}, t_0)\right], \ \forall\, t \in]t_0, t_1], \ \forall\, \mathbf{z} \in S_t, \\ &q_{\{\mathbf{Z}\}}(\mathbf{z}, t_0|\mathbf{x}, t_0) = \delta(\mathbf{z} - \mathbf{x}), \ \forall\, \mathbf{z} \in S_{t_0}, \\ &q_{\{\mathbf{Z}\}}(\mathbf{z}, t|\mathbf{x}, t_0) = 0, \ \forall\, t \in]t_0, t_1], \\ &\mathbf{z} \in \partial S_t^{(0)} \cup \partial S_t^{(2)} \end{aligned}\right\} \tag{7.28}$$

where $\mathcal{K}_{\mathbf{z},t}[\cdots]$ is the forward integro-differential Chapman-Kolmogorov operator (2.26). No boundary condition needs to be formulated on the exit part of the boundary $\partial S_t^{(2)}$.

Knowing the solution of (7.28), the first passage time distribution function, $F_T(t|\mathbf{x}, t_0)$ on condition of a deterministic start in $\mathbf{x} \in S_{t_0}$ is given as

$$F_T(t|\mathbf{x}, t_0) = \Pr\{T < t - t_0\} = 1 - \int_{S_t} q_{\{\mathbf{Z}\}}(\mathbf{z}, t|\mathbf{x}, t_0) d\mathbf{z} \tag{7.29}$$

Let $f_{\{\mathbf{Z}\}}(\mathbf{x}, t_0)$ be the 1st order probability function of the state vector $\{\mathbf{Z}(t), t \in [t_0, t_1]\}$ at the time t_0. The probability density function at the time t on condition of being in the safe domain at the time t_0 is then given as

$$f_{\{\mathbf{Z}\}}(\mathbf{z}, t|S_{t_0}) = \frac{\int_{S_{t_0}} q_{\{\mathbf{Z}\}}(\mathbf{z}, t|\mathbf{x}, t_0) f_{\{\mathbf{Z}\}}(\mathbf{x}, t_0) d\mathbf{x}}{\int_{S_{t_0}} f_{\{\mathbf{Z}\}}(\mathbf{x}, t_0) d\mathbf{x}} \tag{7.30}$$

Since $\mathcal{K}_{\mathbf{z},t}[\cdots]$ is a linear operator, it follows from (7.28) that $f_{\{\mathbf{Z}\}}(\mathbf{z}, t|S_{t_0})$ fulfills the

boundary and initial value problem

$$\left.\begin{aligned}
&\frac{\partial}{\partial t} f_{\{\mathbf{Z}\}}(\mathbf{z}, t \mid S_{t_0}) = \mathcal{K}_{\mathbf{z},t}\left[f_{\{\mathbf{Z}\}}(\mathbf{z}, t \mid S_{t_0})\right],\ \forall\, t \in]t_0, t_1],\ \forall\, \mathbf{z} \in S_t \\
&f_{\{\mathbf{Z}\}}(\mathbf{z}, t_0 \mid S_{t_0}) = \frac{f_{\{\mathbf{Z}\}}(\mathbf{x}, t_0)}{\int\limits_{S_{t_0}} f_{\{\mathbf{Z}\}}(\mathbf{x}, t_0) d\mathbf{x}},\ \forall\, \mathbf{z} \in S_{t_0} \\
&f_{\{\mathbf{Z}\}}(\mathbf{z}, t_0 \mid S_{t_0}) = 0,\ \forall\, t \in]t_0, t_1] \\
&\mathbf{z} \in \partial S_t^{(0)} \cup \partial S_t^{(2)}
\end{aligned}\right\} \tag{7.31}$$

The first passage time distribution function on condition of stochastic start at the time t_0 is then given as

$$F_T(t|S_{t_0}) = 1 - \int\limits_{S_t} f_{\{\mathbf{Z}\}}(\mathbf{z}, t \mid S_{t_0}) d\mathbf{z} \tag{7.32}$$

Alternatively, the reliability problem can be formulated based on the backward integro-differential Chapman-Kolmogorov equation. In order to formulate the boundary conditions for absorbtion of sample curves it is noticed that any state $\mathbf{z} \in \partial S_t^{(1)}$ on the exit part of the boundary inevitably leads to an out-crossing into the unsafe domain. Hence, it is not possible to have a state $\mathbf{z} \in \partial S_t^{(1)}$, and a state $\mathbf{y} \in S_{t_1}$ in the safe domain at the later time t_1, without performing one or more out-crossings into the unsafe domain in the intermediate interval $]t, t_1]$. Since we are interested in the sample curves, which remain in the safe domain throughout the interval $]t, t_1]$, transitions from $\mathbf{z} \in \partial S_t^{(1)}$ to $\mathbf{y} \in S_{t_1}$ should then be prevented. Hence, $q_{\{\mathbf{Z}\}}(\mathbf{y}, t_1|\mathbf{z}, t)$ should fulfil the following boundary and terminal value problem, cf. (2.46), (2.48)

$$\left.\begin{aligned}
&\frac{\partial}{\partial t} q_{\{\mathbf{Z}\}}(\mathbf{y}, t_1|\mathbf{z}, t) + \mathcal{K}^T_{\mathbf{z},t}\left[q_{\{\mathbf{Z}\}}(\mathbf{y}, t_1 \mid \mathbf{z}, t)\right] = 0,\ \forall\, t \in [t_0, t_1[,\ \forall\, \mathbf{z} \in S_t \\
&q_{\{\mathbf{Z}\}}(\mathbf{y}, t_1|\mathbf{z}, t_1) = \delta(\mathbf{z} - \mathbf{y}),\ \forall\, \mathbf{z} \in S_{t_1} \\
&q_{\{\mathbf{Z}\}}(\mathbf{y}, t_1|\mathbf{z}, t) = 0,\ \forall\, t \in]t_0, t_1[\\
&\mathbf{z} \in \partial S_t^{(1)} \cup \partial S_t^{(2)}
\end{aligned}\right\} \tag{7.33}$$

where $\mathcal{K}^T_{\mathbf{z},t}[\cdots]$ is the backward integro-differential Chapman-Kolmogorov operator (2.40). No boundary condition needs to be formulated on the entrance part of the boundary $S_t^{(0)}$. In combination the absorbtion boundary conditions in (7.28) and (7.31) fulfill the necessary boundary condition (2.49).

Upon inserting into (7.31), the first-passage time distribution function $F_T(t_1|\mathbf{z}, t)$ at the time t_1 on condition of the deterministic start in $\mathbf{z} \in S_t$ as defined by (7.29) is seen to

fulfil the following boundary and terminal value problem

$$\left.\begin{aligned} &\frac{\partial}{\partial t}F_T(t_1|\mathbf{z},t) + \mathcal{K}^T_{\mathbf{z},t}\left[F_T(t_1 \mid \mathbf{z},t)\right] = 0,\ \forall\, t \in [t_0,t_1[,\ \forall\, \mathbf{z} \in S_t \\ &F_T(t_1|\mathbf{z},t_1) = 0,\ \forall\, \mathbf{z} \in S_{t_1} \\ &F_T(t_1|\mathbf{z},t) = 1,\ \forall\, t \in]t_0,t_1[\\ &\mathbf{z} \in \partial S_t^{(1)} \cup \partial S_t^{(2)} \end{aligned}\right\} \tag{7.34}$$

Assume that (7.34) is integrated backward until the time $t = t_0$. From the obtained solution $F_T(t_1|\mathbf{z},t_0)$, the first-passage time probability distribution function at the time t_1 on condition of stochastic start at the time t_0 is then obtained from

$$F_T(t_1 \mid S_{t_0}) = \frac{\int\limits_{S_{t_0}} F_T(t_1 \mid \mathbf{z},t_0) f_{\{\mathbf{Z}\}}(\mathbf{z},t_0)d\mathbf{z}}{\int\limits_{S_{t_0}} f_{\{\mathbf{Z}\}}(\mathbf{z},t_0)d\mathbf{z}} \tag{7.35}$$

A significant facilitation is obtained, when the following stationarity conditions are fulfilled

$$q_{\{\mathbf{Z}\}}(\mathbf{y},t_1 \mid \mathbf{z},t) = q_{\{\mathbf{Z}\}}(\mathbf{y} \mid \mathbf{z},t_1 - t) \tag{7.36}$$

$$S_t \equiv S \qquad \text{(time-invariant)} \tag{7.37}$$

Equation (7.32) will be fulfilled, if the generating source processes $\{\mathbf{W}(t), t \in]t_0,t_1]\}$ and $\{\mathbf{V}(t), t \in]t_0,t_1]\}$ are stationary, and if the structural system is time invariant, i.e. if the drift vector and the diffusion matrices fulfil $\mathbf{c}\big(\mathbf{Z}(t),t\big) \equiv \mathbf{c}\big(\mathbf{Z}(t)\big)$ and $\mathbf{b}\big(\mathbf{Z}(t),t\big) \equiv \mathbf{b}\big(\mathbf{Z}(t)\big)$ and $\mathbf{d}\big(\mathbf{Z}(t),t\big) \equiv \mathbf{d}\big(\mathbf{Z}(t)\big)$.

Then, the forward and backward Kolmogorov operators do not explicitly depend on time, i.e. $\mathcal{K}_{\mathbf{z},t}[\cdots] \equiv \mathcal{K}_{\mathbf{z}}[\cdots]$ and $\mathcal{K}^T_{\mathbf{z},t}[\cdots] \equiv \mathcal{K}^T_{\mathbf{z}}[\cdots]$. From (7.29) it follows that $F_T(t_1 \mid \mathbf{z},t) \equiv F_T(\tau \mid \mathbf{z})$, where $\tau = t - t_1$ signifies the elapsed time interval. Equation (7.34) can then be reformulated in the following way

$$\left.\begin{aligned} &\frac{\partial}{\partial \tau}F_T(\tau \mid \mathbf{z}) - \mathcal{K}^T_{\mathbf{z}}\left[F_T(\tau \mid \mathbf{z})\right] = 0,\ \forall\, \tau \in [0,\infty[,\ \forall\, \mathbf{z} \in S \\ &F_T(0 \mid \mathbf{z}) = 0,\ \forall\, \mathbf{z} \in S \\ &F_T(\tau \mid \mathbf{z}) = 1,\ \forall\, \tau \in]0,\infty[\ ,\ \mathbf{z} \in \partial S^{(1)} \cup \partial S^{(2)} \end{aligned}\right\} \tag{7.38}$$

(7.34) must be solved for each terminal time t_1 to get (7.35), whereas (7.38) only requires a single solution of the same initial and boundary problem to obtain all terminal times. Finally, $F_T(t \mid S_{t_0})$ can be obtained from the solution of (7.38) as follows, cf. (7.35)

$$F_T(t \mid S_{t_0}) = \frac{\int\limits_{S} F_T(t - t_0 \mid \mathbf{z}) f_{\{\mathbf{Z}\}}(\mathbf{z})d\mathbf{z}}{\int\limits_{S} f_{\{\mathbf{Z}\}}(\mathbf{z})d\mathbf{z}} \tag{7.39}$$

$F_T(\tau \mid \mathbf{z})$, $\tau = t - t_0$, as determined from (7.38) specifies the first-passage probability distribution function in case of deterministic start in the state $\mathbf{Z}(t_0) = \mathbf{z} \in S$ at the time t_0. From (7.38) it follows that the Nth order moment of the first-passage time, $m_N(\mathbf{z}) = E\left[T^N \mid \mathbf{Z}(t_0) = \mathbf{z} \in S\right]$, $N = 1, 2, \ldots$, can be obtained from the following recursive system of boundary value problems

$$\left.\begin{aligned} &N m_{N-1}(\mathbf{z}) + \mathcal{K}_{\mathbf{z}}^T\left[m_N(\mathbf{z})\right] = 0, \ \forall\, \mathbf{z} \in S, N = 1, 2, \ldots \\ &m_N(\mathbf{z}) = 0, \ \forall\, \mathbf{z} \in \partial S^{(1)} \cup \partial S^{(2)} \end{aligned}\right\} \tag{7.40}$$

where $m_0(\mathbf{z}) \equiv 1$. The case $N = 1$ represents the classical Andronov-Pontriagin-Vitt equation. The general equation (7.40) is known as the generalized Andronov-Pontriagin-Vitt equation, Andronov, Pontriagin and Vitt (1933) [7.3], Bolotin (1967) [7.4].

The solution of (7.38) is given by the uniformly convergent series

$$F_T(\tau \mid \mathbf{z}) = 1 - \sum_{n=1}^{\infty} d_n e^{-\lambda_n \tau} \Phi^{(n)}(\mathbf{z}) \tag{7.41}$$

$$d_n = \frac{\int\limits_S \Psi^{(n)}(\mathbf{z}) d\mathbf{z}}{\int\limits_S \Psi^{(n)}(\mathbf{z}) \Phi^{(n)}(\mathbf{z}) d\mathbf{z}} \tag{7.42}$$

where $\Psi^{(n)}(\mathbf{z})$ and $\Phi^{(n)}(\mathbf{z})$ are the eigenfunctions of the forward and backward operators with the appropriate absorbing boundary conditions, and λ_n are the corresponding eigenvalues, which are all assumed to be single. The indicated quantities are determined from the eigenvalue problems

$$\left.\begin{aligned} &\lambda_n \Psi^{(n)}(\mathbf{z}) + \mathcal{K}_{\mathbf{z}}\left[\Psi^{(n)}(\mathbf{z})\right] = 0, \ \forall\, \mathbf{z} \in S, \ n = 1, 2, \ldots \\ &\Psi^{(n)}(\mathbf{z}) = 0, \ \forall\, \mathbf{z} \in \partial S^{(0)} \cup \partial S^{(2)} \end{aligned}\right\} \tag{7.43}$$

$$\left.\begin{aligned} &\lambda_n \Phi^{(n)}(\mathbf{z}) + \mathcal{K}_{\mathbf{z}}^T\left[\Phi^{(n)}(\mathbf{z})\right] = 0, \ \forall\, \mathbf{z} \in S, \ n = 1, 2, \ldots \\ &\Phi^{(n)}(\mathbf{z}) = 0, \ \forall\, \mathbf{z} \in \partial S^{(1)} \cup \partial S^{(2)} \end{aligned}\right\} \tag{7.44}$$

The solutions $\left(\lambda_n, \Phi^{(n)}\right)$ and $\left(\lambda_n, \Psi^{(n)}\right)$ may be complex. If so, the said solutions appear as pairwise mutually complex conjugated. Further the eigenspectrum has been assumed to be discrete. In the case of a continuous spectrum, (7.41) is replaced by

$$F_T(\tau \mid \mathbf{z}) = 1 - \int_0^{\infty} d(\lambda) e^{-\lambda \tau} \Phi(\mathbf{z}, \lambda) d\lambda \tag{7.45}$$

From (7.43) and (7.44) the following solution can be obtained for the probability density function and the probability distribution function of the first-passage time

$$f_T(t \mid S_{t_0}) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n (t - t_0)} \tag{7.46}$$

$$F_T(t \mid S_{t_0}) = 1 - \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n} e^{-\lambda_n (t-t_0)} \tag{7.47}$$

$$c_n = \lambda_n \frac{\int\limits_S \Psi^{(n)}(\mathbf{z}) d\mathbf{z}}{\int\limits_S \Psi^{(n)}(\mathbf{z}) \Phi^{(n)}(\mathbf{z}) d\mathbf{z}} \frac{\int\limits_S \Phi^{(n)}(\mathbf{z}) f_{\{\mathbf{Z}\}}(\mathbf{z}) d\mathbf{z}}{\int\limits_S f_{\{\mathbf{Z}\}}(\mathbf{z}) d\mathbf{z}} \tag{7.48}$$

The numerical solution of the initial and boundary value problems (7.28), (7.31), (7.38), (7.40), as well as of the eigenvalue problems (7.43), (7.44) involves primarily a discretization of the forward and of the backward Kolmogorov operators. This problem will be reviewed in chapters 8 and 9.

Consider the non-linear SDOF oscillator (1.86) exposed to a Gaussian white noise with the auto-spectral density S_0, and assume that the inertial forces $m\ddot{Y}$ are negligible compared to the other terms entering the equation. Further it is assumed that $g(Y, \dot{Y}) = k(Y)\dot{Y}$. The stochastic equations of motion can then be written as

$$\left.\begin{aligned} & dZ(t) = -c(Z(t))dt + d(Z(t))dW(t), \quad Z(0) = y_0 \\ & Z(t) = Y(t) \ , \ c(Z(t)) = \frac{1}{k(Z)} \frac{\partial}{\partial Z} U(Z) \ , \ d(Z(t)) = \frac{\sqrt{2\pi S_0}}{k(Z)} \end{aligned}\right\} \tag{7.49}$$

where $\{W(t), t \in [0, \infty[\}$ is a unit intensity white noise, cf. section 1.1.1, and y_0 is the initial value.

The drift and diffusion functions become, cf. (2.53), (2.54)

$$C(z) = -c(z) \quad , \quad D(z) = d^2(z) \tag{7.50}$$

In the case of a double barrier first-passage time problem, (7.38) can be written as

$$\left.\begin{aligned} & \frac{\partial}{\partial \tau} F_T(\tau \mid z) - C(z) \frac{\partial}{\partial z} F_T(\tau \mid z) - \frac{D(z)}{2} \frac{\partial^2}{\partial z^2} F_T(\tau \mid z) = 0 \\ & \forall \ \tau \in]0, \infty[\quad , \quad \forall \ z \in]a, b[\\ & F_T(0 \mid z) = 0, \ \forall \ z \in]a, b[\\ & F_T(\tau \mid z) = 1, \ \forall \ \tau \in]0, \infty[\quad , \quad z = a \vee z = b \end{aligned}\right\} \tag{7.51}$$

and (7.43), (7.44) become

$$\left.\begin{aligned} & \lambda_n \Psi^{(n)}(z) - \frac{d}{dz}\left(C(z)\Psi^{(n)}(z)\right) + \frac{1}{2}\frac{d^2}{dz^2}\left(D(z)\Psi^{(n)}(z)\right) = 0 \\ & z \in]a, b[\quad , \quad n = 1, 2, \ldots \\ & \Psi^{(n)}(a) = \Psi^{(n)}(b) = 0 \end{aligned}\right\} \tag{7.52}$$

$$\left.\begin{aligned} &\lambda_n \Phi^{(n)}(z) + C(z)\frac{d}{dz}\Phi^{(n)}(z) + \frac{D(z)}{2}\frac{d^2}{dz^2}\Phi^{(n)}(z) = 0 \ , \ z \in]a,b[, \ n = 1,2,\ldots \\ &\Phi^{(n)}(a) = \Phi^{(n)}(b) = 0 \end{aligned}\right\} \quad (7.53)$$

The eigenvalues λ_n obtained from (7.52) and (7.53) will be well-separated, i.e. the eigenspectrum is discrete. From the solution of (7.52), (7.53) the first-passage time probability distribution function on condition of deterministic start in $y_0 \in]a,b[$ becomes, cf. (7.41), (7.42)

$$F_T(t - t_0 \mid y_0) = 1 - \sum_{n=1}^{\infty} d_n \Phi^{(n)}(y_0) e^{-\lambda_n (t-t_0)} \quad (7.54)$$

$$d_n = \frac{\int_a^b \Psi^{(n)}(z)dz}{\int_a^b \Psi^{(n)}(z)\Phi^{(n)}(z)dz} \quad (7.55)$$

The single barrier problem is obtained as $a \to -\infty$. In this case it is observed, that the separation $\Delta\lambda_n = \lambda_{n+1} - \lambda_n$ approaches zero, i.e. a continuous eigenspectrum is obtained. In this case the discrete eigenvalue expansion (Fourier expansion) is replaced by a continuous eigenvalue integral transform (Fourier transform).

Example 7.2: Brownian motion

As an example consider a linear SDOF oscillator, where also the linear elastic restoring forces are negligible $m2\zeta\omega_0|\dot{Y}| >> m|\ddot{Y}|$, $m2\zeta\omega_0|\dot{Y}| >> m\omega_0^2|Y|$. In this case $k(Z) = m2\zeta\omega_0$, $c(Z) = 0$, so (7.49) reduces to

$$dZ(t) = \frac{1}{m2\zeta\omega_0} dW(t) \quad , \quad Z(0) = y_0 \quad (7.56)$$

This case is the classical Brownian motion equation, studied by Einstein (1905) [7.5]. The drift and diffusion constants become, cf. (7.50)

$$C = 0 \quad , \quad D = \frac{2\pi S_0}{(m2\zeta\omega_0)^2} \quad (7.57)$$

The eigenvalues, eigenfunctions and expansion coefficients become, cf. (7.52), (7.53), (7.55)

$$\left.\begin{aligned} &\lambda_n = \left(\frac{n\pi}{b-a}\right)^2 \frac{D}{2} \\ &\Phi^{(n)}(z) = \Psi^{(n)}(z) = \sin\left(\sqrt{\frac{2\lambda_n}{D}}(b-z)\right) \\ &d_n = \frac{\int_a^b \Phi^{(n)}(z)dz}{\int_a^b \left(\Phi^{(n)}(z)\right)^2 dz} = \frac{2}{n\pi}(1-(-1)^n). \end{aligned}\right\} \quad (7.58)$$

With initial start at $z = y_0$ the solution (7.54) becomes

$$F_T(t - t_0 \mid y_0) = 1 - \sum_{n=1}^{\infty} \frac{2}{n\pi}(1 - (-1)^n) \sin\left(\sqrt{\frac{2\lambda_n}{D}}(b - y_0)\right) e^{-\lambda_n(t-t_0)} \tag{7.59}$$

The difference between the eigenvalues becomes

$$\Delta\lambda_n = \lambda_{n+1} - \lambda_n = \frac{2n+1}{(b-a)^2}\frac{\pi^2 D}{2} \tag{7.60}$$

For the single barrier problem, obtained as $a \to -\infty$, $\Delta\lambda_n$ approaches zero and the continuous eigenspectrum is obtained. Hence, the solution becomes, cf.(7.45)

$$F_T(t - t_0 \mid y_0) = 1 - \int_0^{\infty} d(\lambda) \sin\left(\sqrt{\frac{2\lambda}{D}}(b - y_0)\right) e^{-\lambda(t-t_0)} d\lambda \tag{7.61}$$

Applying the initial value $F_T(0|y_0) = 0$ one has

$$1 = \int_0^{\infty} d(\lambda) \sin\left(\sqrt{\frac{2\lambda}{D}}(b - y_0)\right) d\lambda \tag{7.62}$$

Multiplying by $\sin\left(\sqrt{\frac{2\mu}{D}}(b - y_0)\right)$ and integrating over $[0, \infty[$, $d(\lambda)$ is finally obtained as the following inverse sine Fourier transform

$$d(\lambda) = \frac{1}{\pi}\frac{1}{\lambda} \tag{7.63}$$

The solutions (7.61), (7.63) can next be shown to have the following closed form representation

$$F_T(t - t_0 \mid y_0) = 2 - 2\Phi\left(\frac{b - y_0}{\sqrt{D(t - t_0)}}\right) \tag{7.64}$$

The validity of (7.64) can also be proved directly upon insertion into (7.51).

In case of a discrete eigenspectrum it follows from (7.46) that $f_T(t|S_{t_0}) \propto c_1 e^{-\lambda_1(t-t_0)}$, i.e. the first-passage time probability density function has an asymptotic exponential decay as $t \to \infty$. *The limiting decay rate*, λ_1, forms the lowest eigenvalue of the forward and backward integro-differential Chapman-Kolmogorov operators with absorbing boundary condition. On the other hand, the existence of a limiting decay rate of $f_T(t|S_{t_0})$ is an indication of a discrete eigenspectrum.

The so-called *decay rate coefficient* is defined as

$$\alpha = \frac{\lambda_1}{f_1} \tag{7.65}$$

where f_1 is the out-crossing rate. A running estimate $\alpha(t)$, is next defined from the estimated values $f_T(t|S_{t_0})$ and the value 1 period ahead, assuming $f_T(t|S_{t_0}) = c_1 e^{-\lambda_1(t-t_0)}$. For the simulation results shown in fig. 7.4 the corresponding decay rate coefficient is shown in fig. 7.6. As seen a stationary estimate $\alpha \sim 0.2$ is attained for $t > 10T_0$.

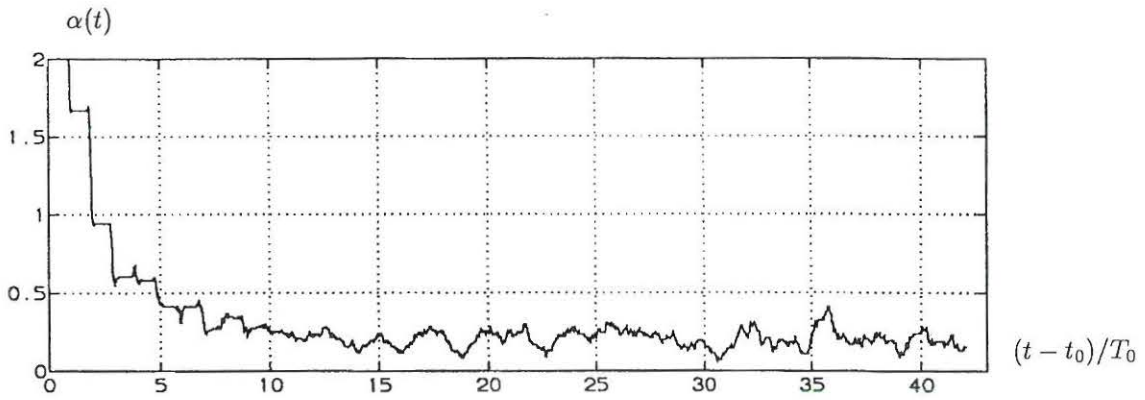

Fig. 7.6. Decay rate coefficient. Single barrier stochastic start problem. SDOF system $\zeta = 0.01$, $b = 2.0\sigma_{Y,0}$.

7.3 Crossing theory

The 1st order out-crossing rate is defined by (7.2). In the same way the nth order crossing rate can be defined as

$$\Pr\left(\bigcap_{i=1}^{n}\{\text{out-crossing in}]t_i, t_i + \Delta t_i]\}\right) = f_n(t_1, \ldots, t_n)\Delta t_1 \cdots \Delta t_n + \mathrm{O}\big((\Delta t_{\max})^{n+1}\big) \quad (7.66)$$

where $\Delta t_{\max} = \max(t_1, \ldots, t_n)$ and the considered intervals are all non-overlapping. $f_n(t_1, \ldots, t_n)$ termed as the nth order crossing rate is the nth degree product density of the point process of out-crossing events. From its definition it follows that $f_n(t_1, \ldots, t_n)$ fulfils the symmetry property

$$f_n(t_1, \ldots, t_n) = f_n(t_{\alpha_1}, \ldots, t_{\alpha_n}) \quad (7.67)$$

where $\alpha_1, \ldots, \alpha_n$ is an arbitrary permutation of $1, \ldots, n$. If the crossing events in the indicated intervals are independent, it follows that

$$\Pr\left(\bigcap_{i=1}^{n}\{\text{out-crossing in}]t_i, t_i + \Delta t_i]\}\right) =$$

$$\prod_{i=1}^{n}\Pr\left(\{\text{out-crossing in}]t_i, t_i + \Delta t_i]\}\right) = \prod_{i=1}^{n}\left(f_1(t_i)\Delta t_i + \mathrm{O}(\Delta t_i^2)\right) \Rightarrow$$

$$f_n(t_1, \ldots, t_n) = \prod_{i=1}^{n} f_1(t_i) \tag{7.68}$$

(7.2) and (7.66) consider out-crossings, independently of whether they originated from the safe domain S_{t_0} at the time t_0 or not. Occasionally, one may also be interested in out-crosssing rates, $f_1(t|S_{t_0})$ and $f_n(t_1, \ldots, t_n|S_{t_0})$, on condition of start in the safe domain at the time t_0. These are defined from the conditional crossing events

$$\Pr\left(\{\text{out-crossing in}]t, t + \Delta t]|\mathbf{Z}(t_0) \in S_{t_0}\}\right) = f_1(t|S_{t_0})\Delta t + \mathrm{O}(\Delta t^2) \tag{7.69}$$

$$\Pr\left(\bigcap_{i=1}^{n}\{\text{out-crossing in}]t_i, t_i + \Delta t_i]|\mathbf{Z}(t_0) \in S_{t_0}\}\right) =$$

$$f_n(t_1, \ldots, t_n|S_{t_0})\Delta t_1 \cdots \Delta t_n + \mathrm{O}\left((\Delta t_{\max})^{n+1}\right)\} \tag{7.70}$$

Let $\{N(t),\ t \in]t_0, \infty[\}$ and $\{N_0(t),\ t \in]t_0, \infty[\}$ be counting processes, specifying the number of out-crossings in $]t_0, t]$ of all M considered realizations and of those M_0 realizations originating in the safe domain, respectively. It follows from (4.4), (4.9) that these counting processes have the product densities $f_1(t_1), f_2(t_1, t_2), \ldots$ and $\Pr(\mathbf{Z}(t_0) \in S_{t_0})$ $f_1(t|S_{t_0})$, $\Pr(\mathbf{Z}(t_0) \in S_{t_0})\, f_2(t_1, t_2|S_{t_0}), \ldots$, respectively. Especially the counting processes become Poissonian, if (7.68) is fulfilled, cf. (4.20).

It follows, that

$$1 - \Pr(N_0(t) = 0) = P_f(]t_0, t]|\mathbf{Z}(t_0) \in S_{t_0}) \Rightarrow$$

$$F_T(t|S_{t_0}) = P_f(]t_0, t]|\mathbf{Z}(t_0) \in S_{t_0}) = \frac{1}{\Pr(\mathbf{Z}(t_0) \in S_{t_0})}\left(1 - \Pr(N_0(t) = 0)\right) =$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \underbrace{\int_{t_0}^{t} \cdots \int_{t_0}^{t}}_{n\text{-fold}} f_n(t_1, \ldots, t_n|S_{t_0}) dt_1 \cdots dt_n \tag{7.71}$$

where (4.14) has been used. (7.71) is known as the inclusion-exclusion series for the first-passage time probability distribution function. The use of (7.71) is limited, partly because the series is divergent at any truncation as $t \to \infty$, and partly because the conditional out-crossing rates can only be calculated for low order n.

If the crossing events are independent, the memory of the initial states disappears, so the conditional and unconditional out-crossing rates become alike, and (7.68) is valid

$$f_n(t_1, \ldots, t_n|S_{t_0}) = f_n(t_1, \ldots, t_n) = \prod_{i=1}^{n} f_1(t_i) \tag{7.72}$$

From (7.71) and (7.72) it follows that

$$F_T(t|S_{t_0}) = 1 - \exp\left(-\int_{t_0}^{t} f_1(\tau)d\tau\right) \tag{7.73}$$

As indicated by (7.8) the hazard rate is also a conditional out-crossing rate.

In case of independent out-crossings it then follows that $h(t) \sim f_1(t)$. Upon inserting this result into (7.10), (7.73) is obtained again.

Stratonovich [7.6] and Roberts [7.7] have tried to close the inclusion-exclusion series by expanding the conditional out-crossing rates $f_n(t_1, \ldots, t_n|S_{t_0})$ in terms of the conditional out-crossing rates $f_1(t_1|S_{t_0})$ and $f_2(t_1, t_2|S_{t_0})$ of the 1st and 2nd order. However, these attempts have been based on rather weak assumptions and seem to have been motivated primarily in order to be able to evaluate the series in a closed form.

In contrast to $f_T(t)$, the out-crossing frequency $f_1(t)$ can be calculated analytically.

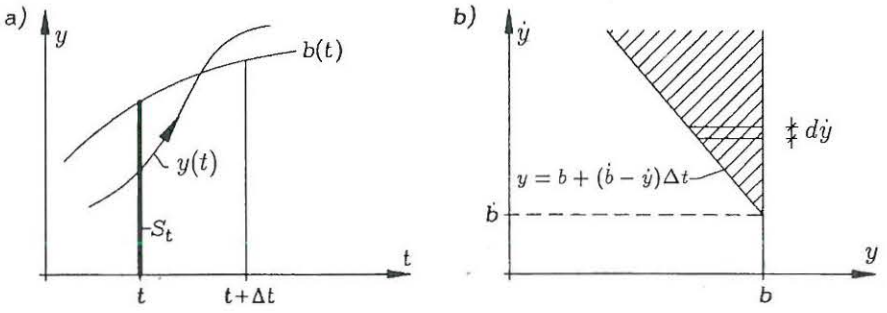

Fig. 7.7. a) Out-crossing in interval $]t, t+\Delta t]$. b) Calculation of out-crossing frequency.

From the definition (7.2) it follows, see fig. 7.7 a)

$$\begin{aligned} &f_1(t)\Delta t + \mathrm{O}(\Delta t^2) = \Pr\big(Y(t) < b(t) \wedge Y(t+\Delta t) \geq b(t+\Delta t)\big) = \\ &\Pr\Big(Y(t) < b(t) \wedge Y(t) + \dot{Y}(t)\Delta t \geq b(t) + \dot{b}(t)\Delta t\Big) = \\ &\Pr\Big(b(t) + \big(\dot{b}(t) - \dot{Y}(t)\big)\Delta t \leq Y(t) < b(t)\Big) \end{aligned} \tag{7.74}$$

where the following 1st order Taylor-expansions $Y(t+\Delta t) = Y(t) + \dot{Y}(t)\Delta t$ and $b(t+\Delta t) = b(t) + \dot{b}(t)\Delta t$ have been applied. Based on the joint pdf $f_{\{Y\}\{\dot{Y}\}}(y, \dot{y}, t)$ of $Y(t)$ and $\dot{Y}(t)$ the right-hand side of (7.74) can be evaluated as follows, see fig. 7.7 b)

$$f_1(t)\Delta t + \mathrm{O}(\Delta t^2) = \int_{\dot{b}}^{\infty} \int_{b+(\dot{b}-\dot{y})\Delta t}^{b} f_{\{Y\}\{\dot{Y}\}}(y, \dot{y}, t)\, dy d\dot{y} =$$

$$\int_{\dot{b}}^{\infty} (\dot{y} - \dot{b})\Delta t \; f_{\{Y\}\{\dot{Y}\}}\Big(b(t) + \theta\big(\dot{b}(t) - \dot{y}\big)\Delta t, \dot{y}, t\Big)\, d\dot{y} \;,\; \theta \in]0,1[\;\Rightarrow$$

$$f_1(t) = \int_{\dot{b}(t)}^{\infty} \big(\dot{y} - \dot{b}(t)\big) f_{\{Y\}\{\dot{Y}\}}\big(b(t), \dot{y}, t\big)\, d\dot{y} \tag{7.75}$$

The 2nd statement of (7.75) follows from the *mean value theorem of integration theory.* (7.75) is termed as *Rice's formula* [7.8].

If b = const. then

$$f_1(t) = \int_{0}^{\infty} \dot{y} f_{\{Y\}\{\dot{Y}\}}(b, \dot{y}, t)\, d\dot{y}$$

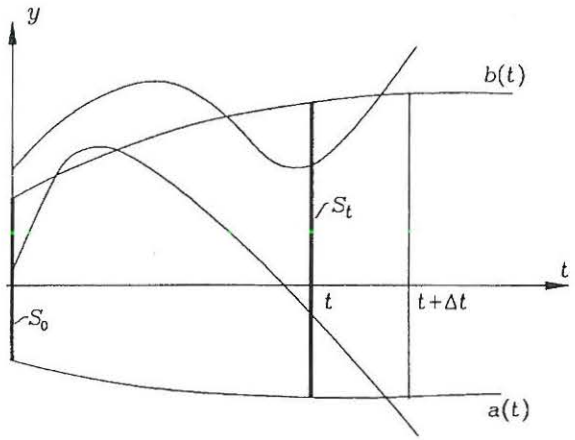

Fig. 7.8. Out-crossings in a double barrier problem.

For a double barrier problem (7.75) generalizes to

$$f_1(t) = \int_{\dot{b}(t)}^{\infty} \big(\dot{y} - \dot{b}(t)\big) f_{\{Y\}\{\dot{Y}\}}\big(b(t), \dot{y}, t\big) d\dot{y} + \int_{-\infty}^{\dot{a}(t)} \big(\dot{a}(t) - \dot{y}\big) f_{\{Y\}\{\dot{Y}\}}\big(a(t), \dot{y}, t\big) d\dot{y} \tag{7.76}$$

(7.76) simply specifies the total out-crossing frequency as the sum of out-crossing frequencies at the upper and at the lower barrier, as illustrated in fig. 7.8.

For a stochastic vector process $\{\mathbf{Y}(t), t \in [0, \infty[\}$ the system is considered safe whenever the realizations are within a differentiable surface Γ_t, which may expand in space. At the

time t a surface element dA_t of Γ_t, specified by the position vector $\mathbf{b}(t)$, is considered. The unit normal vector of the area element in the outward direction is designated $\mathbf{n}\big(\mathbf{b}(t)\big)$, see fig. 7.8. The velocity component in direction of $\mathbf{n}\big(\mathbf{b}(t)\big)$ relative to the failure is then given as

$$\dot{Y}_n(t) = \mathbf{n}^T\big(\mathbf{b}(t)\big)\Big(\dot{\mathbf{Y}}(t) - \dot{\mathbf{b}}(t)\Big) \tag{7.77}$$

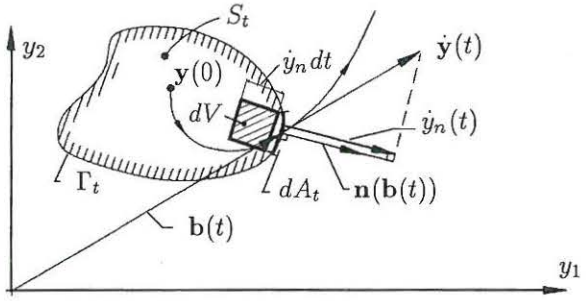

Fig. 7.9. Out-crossings of a stochastic vector process.

where $\{\dot{\mathbf{Y}}(t),\ t \in [t_0, \infty[\}$ is the derivative process. On condition $\dot{Y}_n(t) = \dot{y}_n, \dot{y}_n > 0$, all sample curves in the hatched volume dV of height $\dot{y}_n dt$ will then cross out through the surface element dA_t during the time interval $]t, t + dt]$. Summing up the out-crossing probability contributions from all parts of the failure surface, which are independent events, as a straightforward generalization of (7.76) one has

$$f_1(t)dt = \Pr\big(\{\text{out-crossing in}]t, t + \Delta t]\}\big) =$$

$$\int_{\Gamma_t} \Pr\big(\{\text{out-crossing in}]t, t + \Delta t] \text{ through } dA_t\}\big) =$$

$$\int_{\Gamma_t}\int_0^\infty \Pr\big(\{\text{out-crossing in}]t, t + \Delta t] \text{ through } dA_t | Y_n(t) = \dot{y}_n\}\big)\, f_{\{\dot{Y}_n(t)\}}(\dot{y}_n)d\dot{y}_n \tag{7.78}$$

Since $\Pr\big(\{\text{out-crossing in}]t, t + \Delta t] \text{ through } dA_t | Y_n(t) = \dot{y}_n\}\big) =$
$f_{\{\mathbf{Y}(t)|\dot{Y}_n\}}(\mathbf{y}|\dot{y}_n)dV$, where $dV = \dot{y}_n dt dA_t$ is the hatched volume shown in fig. 7.9 and $\mathbf{y} \simeq \mathbf{b}(t)$ is an interior point in dV one has [7.9]

$$f_1(t)dt = \int_{\Gamma_t}\int_0^\infty f_{\{\mathbf{Y}(t)|\dot{Y}_n\}}(\mathbf{b}(t)|\dot{y}_n)\dot{y}_n dt dA_t f_{\{\dot{Y}_n(t)\}}(\dot{y}_n)d\dot{y}_n \Rightarrow$$

$$f_1(t)dt = \int_{\Gamma_t}\left(\int_0^\infty \dot{y}_n f_{\{\mathbf{Y}\}\{\dot{Y}_n\}}(\mathbf{b}(t), \dot{y}_n, t)\right) dA_t \tag{7.79}$$

Example 7.3: Out-crossing frequency for stationary Gaussian processes

The barrier b is assumed to be constant, and $\{Y(t), t \in R\}$ to be a stationary Gaussian process. Hence, $f_{\{Y\}\{\dot{Y}\}}(y,\dot{y}) = f_{\{Y\}}(y) f_{\{\dot{Y}\}}(\dot{y}) = \frac{1}{\sigma_Y}\varphi\left(\frac{y-\mu_Y}{\sigma_Y}\right)\frac{1}{\sigma_{\dot{Y}}}\varphi\left(\frac{\dot{y}}{\sigma_{\dot{Y}}}\right)$ Since $\dot{b} \equiv 0$, (7.75) becomes

$$f_1 = \int_0^\infty \dot{y}\frac{1}{\sigma_Y}\varphi\left(\frac{b-\mu_Y}{\sigma_Y}\right)\frac{1}{\sigma_{\dot{Y}}}\varphi\left(\frac{\dot{y}}{\sigma_{\dot{Y}}}\right)\,d\dot{y} = \frac{\sigma_{\dot{Y}}}{\sigma_Y}\varphi\left(\frac{b-\mu_Y}{\sigma_Y}\right)\int_0^\infty u\varphi(u)\,du =$$

$$\frac{1}{2\pi}\,\frac{\sigma_{\dot{Y}}}{\sigma_Y}\exp\left(-\frac{1}{2}\left(\frac{b-\mu_Y}{\sigma_Y}\right)^2\right) \tag{7.80}$$

where the integral substitution $u = \frac{\dot{y}}{\sigma_{\dot{Y}}}$ has been performed. Hence, f_1 requires the statistical moments μ_Y, σ_Y and $\sigma_{\dot{Y}}$ to be known. Especially, for $b = \mu_Y$, the expected number of up-crossings ν_0 per unit time of the mean value level is obtained.

Example 7.4: Expected number of local maxima per unit time

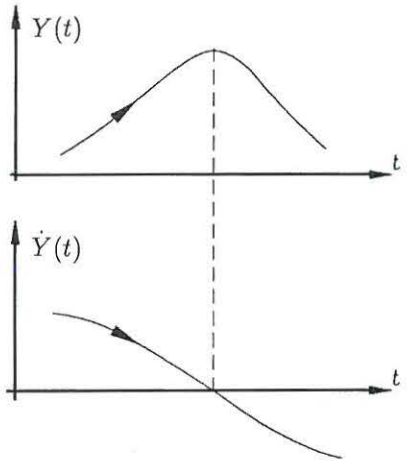

Fig. 7.10. Local maximum of displacement process and simultaneous down-crossing of the velocity process.

A local maximum of the process $\{Y(t), t \in R\}$ is characterized by a down-crossing of the constant barrier $a \equiv 0$ of the velocity process $\{\dot{Y}(t), t \in R\}$, so $\dot{Y}(t) = 0$ at the maximum point, see fig. 7.10. The expected number of local maxima per time $\mu_0(t)$ can then be calculated from (7.75), replacing $[Y(t), \dot{Y}(t)]$ with $[\dot{Y}(t), \ddot{Y}(t)]$, and using $b(t) = \infty$, $a(t) \equiv 0$. Hence

$$\mu_0(t) = -\int_{-\infty}^0 \ddot{y} f_{\{\dot{Y}\}\{\ddot{Y}\}}(0,\ddot{y},t)\,d\ddot{y} \tag{7.81}$$

If $\{Y(t), t \in R\}$ is stationary and Gaussian, then

$$f_{\{\dot{Y}\}\{\ddot{Y}\}}(\dot{y},\ddot{y}) = \frac{1}{\sigma_{\dot{Y}}}\,\varphi\left(\frac{\dot{y}}{\sigma_{\dot{Y}}}\right)\,\frac{1}{\sigma_{\ddot{Y}}}\,\varphi\left(\frac{\ddot{y}}{\sigma_{\ddot{Y}}}\right) \tag{7.82}$$

In this case $\mu_0(t)$ becomes constant. By complete analogy with (7.75) μ_0 is given as

$$\mu_0 = \frac{1}{2\pi}\frac{\sigma_{\dot{Y}}}{\sigma_{\dot{Y}}} \tag{7.83}$$

Example 7.5: Out-crossing frequency from rectangular time-invariant safe domain

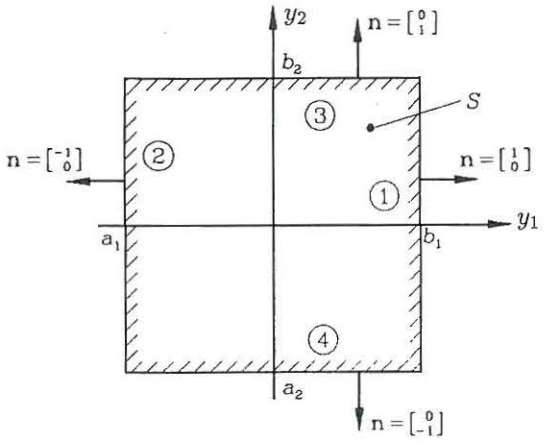

Fig. 7.11. Rectangular safe domain.

The safe domain is given by

$$S = \left\{(y_1, y_2) \mid a_1 < y_1 < b_1 \wedge a_2 < y_2 < b_2\right\} \tag{7.84}$$

where a_1, b_1, a_2, b_2 are time-constant barriers. The out-crossing frequency of the stochastic vector process $\{\mathbf{Y}(t), t \in [0, \infty[\}$, $\mathbf{Y}^T(t) = [Y_1(t), Y_2(t)]$ with respect to the indicated safe domain is to be specified. The surface Γ of S is divided into four sub-surfaces, see fig. 6.11. The out-crossing contributions from each of these four sub-surfaces are determined below.

Sub-surface 1:

$$\left.\begin{aligned} &\mathbf{n} = \begin{bmatrix}1\\0\end{bmatrix}\ ,\ \mathbf{b} = \begin{bmatrix}b_1\\y_2\end{bmatrix}\ ,\ \dot{Y}_n(t) = \mathbf{n}^T\dot{\mathbf{Y}}(t) = \dot{Y}_1(t)\\ &f_{\{\mathbf{Y}\}\{\dot{Y}_n\}}(\mathbf{b}, \dot{y}_n, t) = f_{\{Y_1\}\{Y_2\}\{\dot{Y}_1\}}(b_1, y_2, \dot{y}_n, t)\end{aligned}\right\} \tag{7.85}$$

Sub-surface 2:

$$\left.\begin{aligned} &\mathbf{n} = \begin{bmatrix}-1\\0\end{bmatrix}\ ,\ \mathbf{b} = \begin{bmatrix}a_1\\y_2\end{bmatrix}\ ,\ \dot{Y}_n(t) = \mathbf{n}^T\dot{\mathbf{Y}}(t) = -\dot{Y}_1(t)\\ &f_{\{\mathbf{Y}\}\{\dot{Y}_n\}}(\mathbf{b}, \dot{y}_n, t) = f_{\{Y_1\}\{Y_2\}\{\dot{Y}_1\}}(a_1, y_2, -\dot{y}_n, t)\end{aligned}\right\} \tag{7.86}$$

Sub-surface 3:

$$\left.\begin{aligned} &\mathbf{n} = \begin{bmatrix}0\\1\end{bmatrix}\ ,\ \mathbf{b} = \begin{bmatrix}y_1\\b_2\end{bmatrix}\ ,\ \dot{Y}_n(t) = \mathbf{n}^T\dot{\mathbf{Y}}(t) = \dot{Y}_2(t)\\ &f_{\{\mathbf{Y}\}\{\dot{Y}_n\}}(\mathbf{b}, \dot{y}_n, t) = f_{\{Y_1\}\{Y_2\}\{\dot{Y}_2\}}(y_1, b_2, \dot{y}_n, t)\end{aligned}\right\} \tag{7.87}$$

Sub-surface 4:

$$\left.\begin{aligned} &\mathbf{n} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \ , \ \mathbf{b} = \begin{bmatrix} y_1 \\ a_2 \end{bmatrix} \ , \ \dot{Y}_n(t) = \mathbf{n}^T \dot{\mathbf{Y}}(t) = -\dot{Y}_2(t) \\ &f_{\{\mathbf{Y}\}\{\dot{Y}_n\}}(\mathbf{b}, \dot{y}_n, t) = f_{\{Y_1\}\{Y_2\}\{\dot{Y}_2\}}(y_1, a_1, -\dot{y}_n, t) \end{aligned}\right\} \tag{7.88}$$

In (7.86) and (7.88) the following identity has been applied to the velocity component

$$f_{\{-\dot{Y}\}}(\dot{y}) = f_{\{\dot{Y}\}}(-\dot{y}) \tag{7.89}$$

Using (7.85-88), (7.79) attains the form

$$\begin{aligned} f_1(t) = &\int_{a_2}^{b_2} \left(\int_0^{\infty} \dot{y}_n f_{\{Y_1\}\{Y_2\}\{\dot{Y}_1\}}(b_1, y_2, \dot{y}_n, t)\, d\dot{y}_n \right) dy_2 + \\ &\int_{a_2}^{b_2} \left(\int_0^{\infty} \dot{y}_n f_{\{Y_1\}\{Y_2\}\{\dot{Y}_1\}}(a_1, y_2, -\dot{y}_n, t)\, d\dot{y}_n \right) dy_2 + \\ &\int_{a_1}^{b_1} \left(\int_0^{\infty} \dot{y}_n f_{\{Y_1\}\{Y_2\}\{\dot{Y}_2\}}(y_1, b_2, \dot{y}_n, t)\, d\dot{y}_n \right) dy_1 + \\ &\int_{a_1}^{b_1} \left(\int_0^{\infty} \dot{y}_n f_{\{Y_1\}\{Y_2\}\{\dot{Y}_2\}}(y_1, a_2, -\dot{y}_n, t)\, d\dot{y}_n \right) dy_1 \end{aligned} \tag{7.90}$$

Example 7.6: Out-crossing frequency of stationary Gaussian vector process from time-invariant safe domains

If $\{\mathbf{Y}(t),\ t \in R\}$ is Gaussian, $\{\dot{\mathbf{Y}}(t),\ t \in R\}$ and $\{\dot{Y}_n(t),\ t \in R\}$ become Gaussian, too. The joint probability density $f_{\{\mathbf{Y}\}\{\dot{Y}_n\}}(\mathbf{b}, \dot{y}_n)$ can then be written as

$$f_{\{\mathbf{Y}\}\{\dot{Y}_n\}}(\mathbf{b}, \dot{y}_n) = \frac{1}{\sigma} \varphi\left(\frac{\dot{y}_n - \mu}{\sigma} \right) f_{\{\mathbf{Y}\}}(\mathbf{b}) \tag{7.91}$$

$f_{\{\mathbf{Y}\}}(\mathbf{y})$ is the pdf of an n-dimensional normally distributed stochastic variable. $\mu = \mu(\mathbf{b})$ and $\sigma = \sigma(\mathbf{b})$ signify the expected value and standard deviation of $\dot{Y}_n(t)$ on condition of $\mathbf{Y}(t) = \mathbf{b}$, respectively. Using the results for conditional moments of normal vectors and (7.77) with $E[\dot{\mathbf{Y}}(t)] = \dot{\mathbf{b}}(t) = \mathbf{0}$, these quantities are given by the following expressions

$$\mu(\mathbf{b}) = \mathbf{n}^T(\mathbf{b}) \mathbf{C}_{\mathbf{Y}\dot{\mathbf{Y}}}^T C_{\mathbf{Y}\mathbf{Y}}^{-1} \Big(\mathbf{b} - E[\mathbf{Y}(t)] \Big) \tag{7.92}$$

$$\sigma^2(\mathbf{b}) = \mathbf{n}^T(\mathbf{b}) \Big(\mathbf{C}_{\dot{\mathbf{Y}}\dot{\mathbf{Y}}} - \mathbf{C}_{\mathbf{Y}\dot{\mathbf{Y}}}^T \mathbf{C}_{\mathbf{Y}\mathbf{Y}}^{-1} \mathbf{C}_{\mathbf{Y}\dot{\mathbf{Y}}} \Big) \mathbf{n}(\mathbf{b}) \tag{7.93}$$

where $\mathbf{C}_{\mathbf{Y}\mathbf{Y}}$ and $\mathbf{C}_{\dot{\mathbf{Y}}\dot{\mathbf{Y}}}$ are the covariance matrices of $\mathbf{Y}(t)$ and $\dot{\mathbf{Y}}(t)$, and $\mathbf{C}_{\mathbf{Y}\dot{\mathbf{Y}}} = E[\mathbf{Y}(t)\dot{\mathbf{Y}}^T(t)]$. After inserting (7.91) into (7.79) the following result is obtained

$$f_1 = \int_{\Gamma} \left(\int_0^{\infty} \frac{\dot{y}_n}{\sigma} \varphi\left(\frac{\dot{y}_n - \mu}{\sigma} \right) d\dot{y}_n \right) f_{\{\mathbf{Y}\}}(\mathbf{b})\, dA \tag{7.94}$$

Using integration by parts, the innermost integral of (7.94) can be evaluated as follows

$$\int_0^\infty \frac{\dot{y}_n}{\sigma}\varphi\left(\frac{\dot{y}_n-\mu}{\sigma}\right)d\dot{y}_n = \int_0^\infty \frac{\dot{y}_n-\mu+\mu}{\sigma}\varphi\left(\frac{\dot{y}_n-\mu}{\sigma}\right)d\dot{y}_n =$$

$$\sigma\int_{-\frac{\mu}{\sigma}}^\infty \left(\eta+\frac{\mu}{\sigma}\right)\varphi(\eta)d\eta = \sigma\left[-\varphi(\eta)+\frac{\mu}{\sigma}\Phi(\eta)\right]_{\frac{\mu}{\sigma}}^\infty =$$

$$\sigma\left(\varphi\left(-\frac{\mu}{\sigma}\right)+\frac{\mu}{\sigma}\left(1-\Phi\left(-\frac{\mu}{\sigma}\right)\right)\right) = \sigma\varphi\left(\frac{\mu}{\sigma}\right)+\mu\Phi\left(\frac{\mu}{\sigma}\right) \tag{7.95}$$

where the integration substitution $\eta = \frac{\dot{y}_n-\mu}{\sigma}$ has been applied. Further, the following symmetry properties have been used, $\varphi(y)=\varphi(-y)$, $\Phi(y)=1-\Phi(-y)$. (7.94) can then be written as

$$f_1 = \int_\Gamma \left(\sigma\varphi\left(\frac{\mu}{\sigma}\right)+\mu\Phi\left(\frac{\mu}{\sigma}\right)\right) f_{\{Y\}}(\mathbf{b})\,dA \tag{7.96}$$

Because of the complicated dependency of $\mu=\mu(\mathbf{b})$ and $\sigma=\sigma(\mathbf{b})$ on the coordinates $\mathbf{b}$ of the surface element dA, the integral (7.96) has to be evaluated numerically in most cases. Veneziano *et al.* [7.10] have indicated some analytical results for specialized safe domains and simplified correlation structures as determined by the covariance matrices $\mathbf{C_{YY}}$, $\mathbf{C_{Y\dot{Y}}}$, and $\mathbf{C_{\dot{Y}\dot{Y}}}$.

The results (7.75), (7.76) and (7.79) can be generalized to give the nth order crossings rates. As an example one has for the scalar single barrier case, Andronov et al. [7.3]

$$f_n(t_1,\ldots,t_n) = \underbrace{n-\text{fold}}_{\dot{b}(t_1)} \to \int_{\dot{b}(t_1)}^\infty \cdots \int_{\dot{b}(t_n)}^\infty \left(\dot{y}_1-\dot{b}(t_1)\right)\cdots\left(\dot{y}_n-\dot{b}(t_n)\right)\times$$

$$f_{\{Y\}\{\dot{Y}\}}\left(b(t_1),\dot{y}_1,t_1,\ldots,b(t_n),\dot{y}_n,t_n\right)d\dot{y}_1\cdots d\dot{y}_n \tag{7.97}$$

A further generalization to conditional out-crossing rates $f_1(t|S_{t_0})$ and $f_n(t_1,\ldots,t_n|S_{t_0})$ is obtained upon replacing $f_{\{Y\}\{\dot{Y}\}}\left(y_1,\dot{y}_1,t_1;\ldots;y_n,\dot{y}_n,t_n \mid S_{t_0}\right)$ and $f_{\{\mathbf{Y}\}\{\dot{Y}_n\}}\left(\mathbf{y}_1,\dot{y}_{n,1},t_1;\ldots;\mathbf{y}_n,\dot{y}_{n,n},t_n\right)$ with the nth order joint probability density functions on condition of being in the safe domain at the time t_0 $f_{\{Y\}\{\dot{Y}\}}\left(y_1,\dot{y}_1,t_1;\ldots;y_n,\dot{y}_n,t_n|S_{t_0}\right)$ and $f_{\{\mathbf{Y}\}\{\dot{Y}_n\}}\left(\mathbf{y}_1,\dot{y}_{n,1},t_1;\ldots\right.$ As an example one has in the scalar double barrier case

$$f_{\{Y\}\{\dot{Y}\}}\left(y_1,\dot{y}_1,t_1;\ldots;y_n,\dot{y}_n,t_n|S_{t_0}\right) =$$

$$\frac{\int_{a(t_0)}^{b(t_0)} f_{\{Y\}\{\dot{Y}\}}\left(y_1,\dot{y}_1,t_1;\ldots;y_n,\dot{y}_n,t_n;y_0,t_0\right)dy_0}{\int_{a(t_0)}^{b(t_0)} f_{\{Y\}}(y_0,t_0)dy_0} \tag{7.98}$$

Example 7.7: Unconditional and conditional 1st order out-crossing rates for scalar Gaussian process

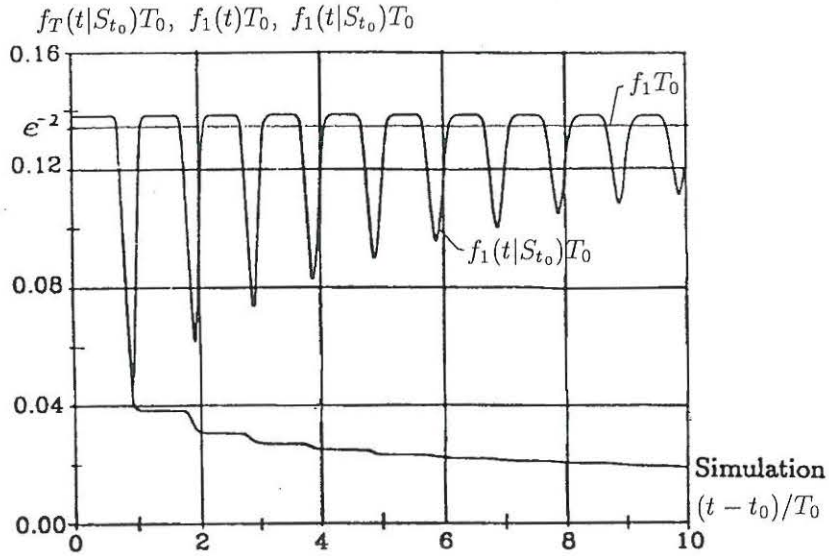

Fig. 7.12. Unconditional and conditional 1st order out-crossing rates. Single barrier stochastic start problem. SDOF system $\zeta = 0.01, b = 2.0\sigma_{Y,0}$.

In fig. 7.12 is shown the conditional out-crossing rate $f_1(t|S_{t_0})$ and the unconditional out-crossing rate $f_1(t) \equiv f_1$, for the first-passage time problem described in relation to fig. 7.4.

In the considered non-dimensional mapping $f_1 \cdot T_0 = e^{-2} = 0.13534$ and $f_1(t_0|S_{t_0}) \cdot T_0 = f_1 \cdot T_0 / \Pr(Y(t_0) \in S_{t_0}) = e^{-2}/\Phi(2) = 0.13849$. As $t \to \infty$ the oscillations of $f_1(t|S_{t_0})$ decrease and $f_1(t|S_{t_0}) \to f_1$.

In fig. 7.12 are also shown the simulation results of fig. 7.4. As seen $f_1(t|S_{t_0})$ forms a true upper bound to $f_T(t|S_{t_0})$. A rigorous proof of this follows from the reformulation, cf. fig. 7.1

$$f_T(t|S_{t_0}) = \frac{\Delta M_1}{\Delta t M_0} = \frac{\Delta M_0}{\Delta t M_0}\frac{\Delta M_1}{\Delta M_0} \leq \frac{\Delta M_0}{\Delta t M_0} = f_1(t|S_{t_0}) \tag{7.99}$$

where the relationship $\Delta M_1 \leq \Delta M_0$ has been used. Upon insertion into (7.5) the following upper bound for the failure probability can be formulated

$$P_f([t_0, t]) \leq 1 - \Pr(\mathbf{Z}(t_0) \in S_{t_0}) + \Pr(\mathbf{Z}(t_0) \in S_{t_0}) \int_{t_0}^{t} f_1(\tau|S_{t_0}) d\tau \tag{7.100}$$

Since $f_1(t|S_{t_0}) = \frac{1}{\Delta t}\frac{\Delta M_0}{M_0} = \frac{1}{\Delta t}\frac{M}{M_0}\frac{\Delta M}{M}\frac{\Delta M_0}{\Delta M}$ and $\Delta M_0 \leq \Delta M$, cf. fig. 7.1, it follows that $f_1(t|S_{t_0}) \leq \frac{1}{\Delta t}\frac{M}{M_0}\frac{\Delta M}{M} = \frac{1}{\Pr(\mathbf{Z}(t_0) \in S_{t_0})} f_1(t)$. It can be then stated, that (7.100) is a sharper upper bound than (7.21).

Below, a single barrier problem with constant upper b is first considered. In this case let $f_{\dot{Y}T}(\dot{y}, t|S_{t_0})$ signify the joint probability density function of the first-passage time T and of the associated out-crossing velocity $\dot{Y}$ on condition of start in S_{t_0}. Using the

Markov property of $\mathbf{Z}^T(t) = [Y(t), \dot{Y}(t)]$, the following integral equation can now be formulated for this quantity, Nielsen [7.11]

$$f_{\dot{Y}T}(\dot{y}, t|S_{t_0}) = \dot{y} f_{\{Y\}\{\dot{Y}\}}(b, \dot{y}, t|S_{t_0}) - \int_{t_0}^{t} \int_{0}^{\infty} \dot{y} q_{\{\mathbf{Z}\}}(b, \dot{y}, t|b, \dot{y}_1, t_1) f_{\dot{Y}T}(\dot{y}_1, t_1|S_{t_0}) d\dot{y}_1 dt_1 \quad (7.101)$$

The first term at the right-hand side of (7.101), i.e. $\dot{y} f_{\{Y\}\{\dot{Y}\}}(b, \dot{y}, t|S_{t_0})$ indicates the rate of out-crossings with out-crossing velocity $\dot{y}$ on condition of start in S_{t_0} at the time t_0. The integral of all these rates of probability densities with respect to $\dot{y}$ provides the conditional 1st order out-crossing rate, $f_1(t|S_{t_0})$, according to (7.75). $f_{\dot{Y}T}(\dot{y}, t|S_{t_0})$ specifies the out-crossings of a subset of sample curves inclusion in $\dot{y} f_{\{Y\}\{\dot{Y}\}}(b, \dot{y}, t|S_{t_0})$, which are also first passing out-crossings. Hence the last term at the right-hand side of (7.74) withdraws from $\dot{y} f_{\{Y\}\{\dot{Y}\}}(b, \dot{y}, t|S_{t_0})$ these out-crossings with the out-crossing velocity $\dot{y}$, which are not first-passages. Indeed, $\dot{y} q_{\{\mathbf{Z}\}}(b, \dot{y}, t|b, \dot{y}_1, t_1) f_{\dot{Y}T}(\dot{y}_1, t_1|S_{t_0})$ specifies the joint rate of out-crossings with the out-crossing velocity $\dot{y}$ at the time t_1 and having a first passage at the previous time $t_1 < t$ with the positive velocity $\dot{y}_1$. Integrating contributions from all possible first-passage times in $]t_0, t]$ and for all possible positive out-crossing velocities $\dot{y}_1$, the rate of out-crossings with out-crossing velocity $\dot{y}$ is obtained.

$f_{\{Y\}\{\dot{Y}\}}(y, \dot{y}, t|S_{t_0})$ signifies the joint probability density of $Y(t)$ and $\dot{Y}(t)$ on condition of $\mathbf{Z}(t_0) \in S_{t_0}$, cf. (7.98), and $q_{\{\mathbf{Z}\}}(y, \dot{y}, t|y_1, \dot{y}_1, t_1)$ is the transition probability density of the Markov process. These quantities are assumed to be known. Then, $f_{\dot{Y}T}(\dot{y}, t|S_{t_0})$ can be found numerically from (7.101). The first-passage time probability density function as a marginal density function is next obtained as follows

$$f_T(t|S_{t_0}) = \int_0^{\infty} f_{\dot{Y}T}(\dot{y}, t|S_{t_0}) d\dot{y} \quad (7.102)$$

Upon integrating (7.101) with respect to $\dot{y}$ the following alternative representation is obtained, using (7.75)

$$f_T(t|S_{t_0}) = f_1(t|S_{t_0}) - \int_{t_0}^{t} \int_{0}^{\infty} f_1(t|b, \dot{y}_1, t_1) f_{\dot{Y}T}(\dot{y}_1, t_1|S_{t_0}) d\dot{y}_1 dt_1 \quad (7.103)$$

where

$$f_1(t|b, \dot{y}_1, t_1) = \int_0^{\infty} \dot{y} q_{\{\mathbf{Z}\}}(b, \dot{y}, t|b, \dot{y}_1, t_1) d\dot{y} \quad (7.104)$$

$f_1(t|b, \dot{y}_1, t_1)$ signifies the out-crossing frequency at the time t on the condition of deterministic start at the time $t_1 < t$ on the boundary with positive out-crossing velocity

$\dot{y}_1 > 0$. At numerical applications (7.103) is more robust than (7.102), since the asymptotic behaviour $f_T(t_0|S_{t_0}) = f_1(t_0|S_{t_0})$ is exactly represented.

Because the integral at the right-hand side of (7.103) is non-negative, the inequality (7.99) follows immediately in this case.

Using the Markov property (2.3) the solution to (7.101) can be expressed in a Neumann-series of iterated kernels

$$f_{\dot{Y}T}(b,\dot{y},t|S_{t_0}) = \dot{y} f_{\{Y\}\{\dot{Y}\}}(b,\dot{y},t|S_{t_0}) -$$

$$\int_{t_0}^{t}\int_{0}^{\infty} \dot{y} q_{\{\mathbf{Z}\}}(b,\dot{y},t \mid b,\dot{y}_1,t_1)\,\dot{y}_1 q_{\{\mathbf{Z}\}}(b,\dot{y}_1,t_1 \mid S_{t_0})\,d\dot{y}_1\,dt_1 + \cdots =$$

$$\dot{y} f_{\{Y\}\{\dot{Y}\}}(b,\dot{y},t \mid S_{t_0}) - \int_{t_0}^{t}\int_{0}^{\infty} \dot{y}\dot{y}_1 f_{\{Y\}\{\dot{Y}\}}(b,\dot{y},t;b,\dot{y}_1,t_1 \mid S_{t_0})\,d\dot{y}_1\,dt_1 + \cdots \quad (7.105)$$

If (7.105) is inserted into (7.102) and (7.75), (7.97) are applied, followed by an integration with respect to time, the inclusion-exclusion series (7.71) is obtained. However, it should be noted that the indicated derivation has only validity for Markov systems, whereas (7.71) is valid even for non-Markovian systems.

For the double barrier problem with time-varying lower and upper barriers $a(t)$ and $b(t)$, (7.101) is replaced with the coupled integral equations

$$f_{\dot{Y}T}(b(t),\dot{y},t \mid S_{t_0}) = (\dot{y} - \dot{b}(t)) f_{\{Y\}\{\dot{Y}\}}(b(t),\dot{y},t \mid S_{t_0}) -$$

$$\int_{t_0}^{t}\int_{\dot{b}(t_1)}^{\infty} (\dot{y} - \dot{b}(t)) q_{\{\mathbf{Z}\}}(b(t),\dot{y},t \mid b(t_1),\dot{y}_1,t_1) f_{\dot{Y}T}(b(t_1),\dot{y}_1,t_1 \mid S_{t_0})\,d\dot{y}_1\,dt_1 -$$

$$\int_{t_0}^{t}\int_{-\infty}^{\dot{a}(t_1)} (\dot{y} - \dot{b}(t)) q_{\{\mathbf{Z}\}}(b(t),\dot{y},t \mid a(t_1),\dot{y}_1,t_1) f_{\dot{Y}T}(a(t_1),\dot{y}_1,t_1 \mid S_{t_0})\,d\dot{y}_1\,dt_1 \quad (7.106)$$

$$f_{\dot{Y}T}(a(t),\dot{y},t \mid S_{t_0}) = (\dot{a}(t) - \dot{y}) f_{\{Y\}\{\dot{Y}\}}(a(t),\dot{y},t \mid S_{t_0}) -$$

$$\int_{t_0}^{t}\int_{-\infty}^{\dot{a}(t_1)} (\dot{a}(t) - \dot{y}) q_{\{\mathbf{Z}\}}(a(t),\dot{y},t \mid a(t_1),\dot{y}_1,t_1) f_{\dot{Y}T}(a(t_1),\dot{y}_1,t_1 \mid S_{t_0})\,d\dot{y}_1\,dt_1 -$$

$$\int_{t_0}^{t}\int_{\dot{b}(t_1)}^{\infty} (\dot{a}(t) - \dot{y}) q_{\{\mathbf{Z}\}}(a(t),\dot{y},t \mid b(t_1),\dot{y}_1,t_1) f_{\dot{Y}T}(b(t_1),\dot{y}_1,t_1 \mid S_{t_0})\,d\dot{y}_1\,dt_1 \quad (7.107)$$

From the solution of (7.106) and (7.107) $f_T(t \mid S_{t_0})$ is then obtained from

$$f_T(t \mid S_{t_0}) = \int_{\dot{b}(t)}^{\infty} f_{\dot{Y}T}\big(b(t), \dot{y}, t \mid S_{t_0}\big) d\dot{y} + \int_{-\infty}^{\dot{a}(t)} f_{\dot{Y}T}\big(a(t), \dot{y}, t \mid S_{t_0}\big)\, d\dot{y} \tag{7.108}$$

Next, consider the vector crossing problem shown in fig. 7.8. Let $f_{\dot{\mathbf{Y}}T}\big(\dot{\mathbf{y}}, t; \mathbf{b}(t) \mid S_{t_0}\big)$ signify the joint probability density function of the first-passage time T and the associated out-crossing velocity $\mathbf{Y}$ per unit area of the failure surface Γ_t at the position $\mathbf{b}(t)$ on condition of start in S_{t_0}. The rate of out-crossings per unit area of the failure surface Γ_t at the position $\mathbf{b}(t)$ with the out-crossing velocity $\mathbf{y}$, and on condition of start in S_{t_0} is given as $\mathbf{n}^T\big(\mathbf{b}(t)\big)\big(\dot{\mathbf{y}} - \dot{\mathbf{b}}(t)\big) f_{\{\mathbf{Y}\}\{\dot{\mathbf{Y}}\}}\big(\mathbf{b}(t), \dot{\mathbf{y}}, t \mid S_{t_0}\big)$, cf. (7.79). This quantity includes all the corresponding first-passage out-crossings, which are considered by $f_{\dot{\mathbf{Y}}T}\big(\dot{\mathbf{y}}, t; \mathbf{b}(t) \mid S_{t_0}\big)$ plus some out-crossings which have had their first-passage prior to the time t_1. The following integral equation generalizing (7.101) and (7.106), (7.107) can then be formulated

$$\begin{aligned} f_{\dot{\mathbf{Y}}T}\big(\dot{\mathbf{y}}, t; \mathbf{b}(t) \mid S_{t_0}\big) &= \mathbf{n}^T\big(\mathbf{b}(t)\big)\big(\dot{\mathbf{y}} - \dot{\mathbf{b}}(t)\big) f_{\{\mathbf{Y}\}\{\dot{\mathbf{Y}}\}}\big(\mathbf{b}(t), \dot{\mathbf{y}}, t \mid S_{t_0}\big) - \\ &\int_{t_0}^{t} \mathbf{n}^T\big(\mathbf{b}(t)\big)\big(\dot{\mathbf{y}} - \dot{\mathbf{b}}(t)\big) \int_{\Gamma_{t_1}} \int_{\Lambda_{t_1}} q_{\{\mathbf{Z}\}}\big(\mathbf{b}(t), \dot{\mathbf{y}}, t \mid \mathbf{b}(t_1), \dot{\mathbf{y}}_1, t_1\big) \times \\ &f_{\dot{\mathbf{Y}}T}\big(\dot{\mathbf{y}}_1, t_1; \mathbf{b}(t_1) \mid S_{t_0}\big)\, d\dot{\mathbf{y}}_1 da_{t_1} dt_1 \end{aligned} \tag{7.109}$$

where

$$\Lambda_t = \{\dot{\mathbf{y}} \in R^n \mid \mathbf{n}^T\big(\mathbf{b}(t)\big)\big(\dot{\mathbf{y}} - \dot{\mathbf{b}}(t)\big) > 0\} \tag{7.110}$$

Λ_t signifies the set of velocities $\dot{\mathbf{y}}$, which implies the out-crossing through the surface element dA_t. (7.109) is related to the Belyaev's formula (7.79) in the same way as (7.101) and (7.106), (7.107) are related to the Rice formula (7.75) and (7.76). (7.109) defines the integral equations for all points $\mathbf{b}(t)$ of the surface Γ_t. In case of numerical solutions the failure surface Γ_t has to be divided into finite subdomains within each of which $f_{\dot{\mathbf{Y}}T}(\dot{\mathbf{y}}, t; \mathbf{b}(t) \mid S_{t_0})$ is considered constant as a function of $\mathbf{b}(t)$. A closed system of integral equations for these quantities may then be obtained from (7.109). Hence, in contrast to (7.101) and (7.106), (7.107) no exact formulation of the problem is obtained in this case. The level of approximation depends on the roughness of the discretization of the failure surface. Obviously, a finer discretization should be applied where out-crossings are likely to occur. The first-passage time probability density function is next obtained from

$$f_T(t \mid S_{t_0}) = \int_{\Gamma_t} \int_{\Lambda_t} f_{\dot{\mathbf{Y}}T}\big(\dot{\mathbf{y}}, t; \mathbf{b}(t) \mid S_{t_0}\big)\, d\dot{\mathbf{y}} dA_t \tag{7.111}$$

Example 7.8: Single barrier stationary start first-passage time problem for single-degree-of-freedom oscillator subjected to Gaussian white noise or to Poisson driven trains of impulses

A linear time-invariant SDOF system is considered subjected to the generating source $\{V(\tau),\ \tau \in [t_0, t_1]\}$ which may be either a stationary Gaussian white noise excitation or a stationary compound Poisson process.

In this case the restoring force in (1.86) can be written as

$$g(Y,\dot{Y}) + \frac{\partial}{\partial Y}U(Y) = m(2\zeta\omega_0\dot{Y}(t) + \omega_0^2 Y(t)) \tag{7.112}$$

The displacement and the velocity of the system on condition of the initial values $\mathbf{z}_0^T = [Y(t_0), \dot{Y}(t_0)] = [y_0, \dot{y}_0]$ at the time t_0 can then be written as

$$Y(t) = c(t \mid \mathbf{z}_0, t_0) + \int_{t_0^+}^{t^-} h(t - t_1)\, dV(t_1) \tag{7.113}$$

$$\dot{Y}(t) = \dot{c}(t \mid \mathbf{z}, t_0) + \int_{t_0^+}^{t^-} \dot{h}(t - t_1)\, dV(t_1) \tag{7.114}$$

$$c(t \mid \mathbf{z}_0, t_0) = (\dot{h}(t - t_0) + 2\zeta\omega_0 h(t - t_0))y_0 + h(t - t_0)\dot{y}_0 \tag{7.115}$$

$$\dot{c}(t \mid \mathbf{z}_0, t_0) = -\omega_0^2 h(t - t_0)y_0 + \dot{h}(t - t_0)\dot{y}_0 \tag{7.116}$$

$$h(t) = \begin{cases} 0 & , \ t < 0 \\ \frac{1}{m\omega_0\sqrt{1-\zeta^2}} e^{-\zeta\omega_0 t} \sin\left(\omega_0\sqrt{1-\zeta^2}t\right) & , \ t \geq 0 \end{cases} \tag{7.117}$$

$h(t)$ and $\dot{h}(t)$ are the impulse response functions of the displacement and the velocity. The functions $c(t \mid \mathbf{z}_0, t_0)$ and $\dot{c}(t \mid \mathbf{z}_0, t_0)$ indicate the deterministic drift (the eigenvibrations) of the displacement and the velocity from the initial value $\mathbf{z}_0$ at the time t_0.

The joint characteristic function of $Y(t)$ and $\dot{Y}(t)$ on condition of $\mathbf{Z}(t_0) = \mathbf{z}_0$ is denoted $\Phi_{\{\mathbf{Z}\}}(\theta_1, \theta_2, t \mid y_0, \dot{y}_0, t_0)$. $\lambda_{mn}[\mathbf{Z}(t) \mid \mathbf{z}_0, t_0] = \frac{\partial^{m+n}}{\partial(i\theta_1)^m \partial(i\theta_2)^n} \ln \Phi_{\{\mathbf{Z}\}}(\theta_1, \theta_2, t \mid y_0, \dot{y}_0, t_0)\,|_{\theta_1=\theta_2=0}$ signifies the joint cumulant of the order $m + n$ of $Y(t)$ and $\dot{Y}(t)$, cf. (4.53).

Since the Wiener process is Gaussian, the response process also becomes Gaussian. The joint transition probability density function of the Markov state vector can then immediately be indicated as follows

$$q_{\{\mathbf{Z}\}}(y, \dot{y}, t \mid y_0, \dot{y}_0, t_0) = \frac{\varphi_2(\xi_1, \xi_2; \rho[\mathbf{Z}(t) \mid \mathbf{z}_0, t_0])}{\sigma[Y(t) \mid \mathbf{z}_0, t_0]\sigma[\dot{Y}(t) \mid \mathbf{z}_0, t_0]} \tag{7.118}$$

$$\xi_1 = \frac{y - c(t \mid \mathbf{z}_0, t_0)}{\sigma[Y(t) \mid \mathbf{z}_0, t_0]} \quad , \quad \xi_2 = \frac{\dot{y} - \dot{c}(t \mid \mathbf{z}_0, t_0)}{\sigma[\dot{Y}(t) \mid \mathbf{z}_0, t_0]} \tag{7.119}$$

$$\lambda_{mn}(\mathbf{Z}(t) \mid \mathbf{z}_0, t_0) = \int_{t_0}^{t} h^m(t - \tau)\dot{h}^n(t - \tau)2\pi S_{t_0}\, d\tau \quad , \quad m + n = 2 \tag{7.120}$$

In this case, $c(t \mid \mathbf{z}_0, t_0)$ and $\dot{c}(t \mid \mathbf{z}_0, t_0)$ can be identified as the conditional mean value functions of the displacement and the velocity processes. $\sigma[Y(t) \mid \mathbf{z}_0, t_0]$, $\sigma[\dot{Y} \mid \mathbf{z}_0, t_0]$ and $\rho[\mathbf{Z}(t) \mid \mathbf{z}_0, t_0]$ signify the conditioned standard deviations and conditioned correlation coefficient function of $Y(t)$ and $\dot{Y}(t)$ as calculated from (7.120). $\varphi_2(\xi_1, \xi_2; \rho)$ is the joint probability density function of a bivariate normal stochastic variable with zero mean values, unit standard deviations and the correlation coefficient ρ. (7.118) was first obtained by Wang and Uhlenbeck [7.12] based on direct integration of the associated Fokker-Planck equation.

A stochastic start problem with a constant upper barrier b is considered. In this case the stationary displacement $Y(0)$ and the stationary velocity $\dot{Y}(0)$ are stochastically independent. The conditional joint probability density function $f_{\{Y\}\{\dot{Y}\}}(b, \dot{y}, t \mid S_{t_0})$ as given by (7.108) then becomes

$$f_{\{Y\}\{\dot{Y}\}}(b, \dot{y}, t \mid S_{t_0}) = \frac{1}{\Phi\left(\frac{b}{\sigma_{Y,0}}\right)} \cdot \int_{-\infty}^{b} \frac{\varphi_2(\xi_1, \xi_2; \rho)}{\sigma[Y(t) \mid \mathbf{z}_0, 0]\, \sigma[\dot{Y}(t) \mid \mathbf{z}_0, 0]} \frac{\varphi\left(\frac{y}{\sigma_{Y,0}}\right)}{\sigma_{Y,0}} dy \tag{7.121}$$

where $\sigma_{Y,0} = \lim_{t \to \infty} \sigma[Y(t) \mid \mathbf{z}_0, 0]$, and $\varphi(y)$ and $\Phi(y)$ are the probability density function and the distribution function of a standardized normal variable. The integral in (7.121) can be evaluated analytically in terms of $\varphi(y)$ and $\Phi(y)$.

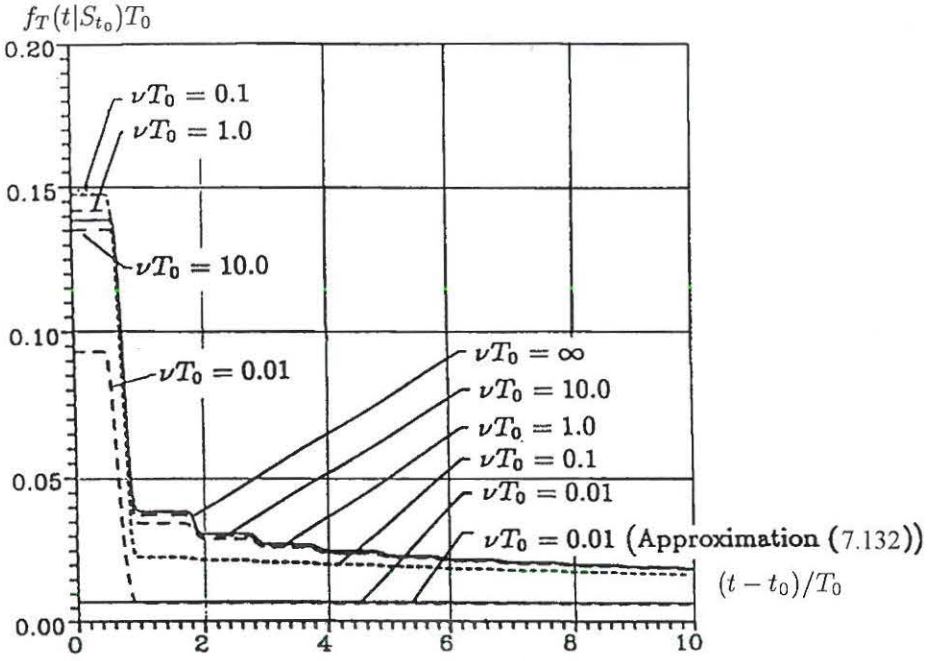

Fig. 7.13: First-passage time probability density functions for white noise and Poisson driven pulses of an SDOF oscillator. Single barrier stochastic start problem $\zeta = 0.01$, $b = 2.0\,\sigma_{Y,0}$. Nielsen [7.11].

Interval $\frac{t}{T_0}$	Exact solution $f_{T_1}(t \mid S_{t_0})T_0$
0–1	0.13849
1–2	0.03821
2–3	0.03041
3–4	0.02678
4–5	0.02453

Table 7.1: Stair levels of first-passage probability density function. SDOF oscillator exposed to Gaussian white noise.

The integral equation (7.101) was solved numerically, using a trapezoidal scheme with the time step length $\Delta t_1 = 0.025T_0$ and a Gaussian quadrature scheme with the velocity step length $\Delta\dot{y}_1 = 0.15\sigma_{\dot{Y},0}$, where $\sigma_{\dot{Y},0} = \lim_{t\to\infty} \sigma[\dot{Y}(t) \mid \mathbf{z}_0, 0]$ is the stationary velocity standard deviation. The latter scheme was also used for the quadrature in (7.103). The result has been shown as the full-line curve $\nu T_0 = \infty$ in fig. 7.13. The horizontal stair levels of the first-passage time probability density function are indicated in table 7.1.

In case of compound Poisson excitation the increment of the excitation process in the interval $]t, t+dt]$ is given by (1.52).

Assuming that the moments $E[P^n]$ of sufficiently high order n exist the conditioned mean values and the joint conditioned cumulants of the response process become

$$\mu[Y(t) \mid \mathbf{z}_0, t_0] = c(t \mid \mathbf{z}_0, t_0) + \int_{t_0}^{t} h(t-\tau)\nu(\tau)E[P]d\tau \tag{7.122}$$

$$\mu[\dot{Y}(t) \mid \mathbf{z}_0, t_0] = \dot{c}(t \mid \mathbf{z}_0, t_0) + \int_{t_0}^{t} \dot{h}(t-\tau)\nu(\tau)E[P]d\tau \tag{7.123}$$

$$\lambda_{mn}[\mathbf{Z}(t) \mid \mathbf{z}_0, t_0] = \int_{t_0}^{t} h^m(t-\tau)\dot{h}^n(t-\tau)\nu(\tau)E[P^{m+n}]d\tau \quad , \quad m+n \geq 2 \tag{7.124}$$

The integrals in (7.122), (7.123), (7.124) can all be evaluated analytically in case of stationary impulse rates ν. From (7.122), (7.123), (7.124) it is seen that the log-characteristic function of the conditioned response process is given by

$$\ln \Phi_{\{\mathbf{Z}\}}(\theta_1, \theta_2, t \mid y_0, \dot{y}_0, t_0) = c(t \mid \mathbf{z}_0, t_0)i\theta_1 + \dot{c}(t \mid \mathbf{z}_0, t_0)i\theta_2 + S(i\theta_1, i\theta_2) \tag{7.125}$$

$$S(i\theta_1, i\theta_2) = \int_{t_0}^{t} \nu(\tau)\Big(E[\exp(Ph(t-\tau)i\theta_1 + P\dot{h}(t-\tau)i\theta_2)] - 1\Big)d\tau \tag{7.126}$$

The joint transition probability density function can then be obtained by a double inverse Fourier transformation

$$q_{\{\mathbf{Z}\}}(y, \dot{y}, t \mid y_0, \dot{y}_0, t_0) =$$

$$\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi_{\{\mathbf{Z}\}}(\theta_1, \theta_2, t \mid y_0, \dot{y}_0, t_0) \exp(-yi\theta_1 - \dot{y}i\theta_2)d\theta_1 d\theta_2 =$$

$$\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\Big(S(i\theta_1, i\theta_2) - (y-c)i\theta_1 - (\dot{y}-\dot{c})i\theta_2\Big)d\theta_1 d\theta_2 \tag{7.127}$$

(7.127) shows that the joint transition probability density depends on $y, \dot{y}, y_0, \dot{y}_0$ through the differences $y - c(t \mid \mathbf{z}_0, t_0)$ and $\dot{y} - \dot{c}(t \mid \mathbf{z}_0, t_0)$. Unfortunately, (7.127) cannot be solved in closed form for any

system of engineering interest. Alternatively, the solution may be represented by the following infinite Gram-Charlier series on Edgeworth form, Longuet-Higgins (1964) [7.13], cf. (3.63)

$$q_{\{\mathbf{Z}\}}(y,\dot{y},t \mid y_0,\dot{y}_0,t_0) = \frac{\varphi_2(\xi_1,\xi_2;\rho[\mathbf{Z}(t) \mid \mathbf{z}_0,t_0])}{\sigma[Y(t) \mid \mathbf{z}_0,t_0]\,\sigma[\dot{Y}(t) \mid \mathbf{z}_0,t_0]} \Bigg(1 +$$

$$\sum_{k+l=3}^{\infty} \frac{\lambda_{kl}^0}{k!\,l!} H_{kl}(\xi_1,\xi_2;\rho[\mathbf{Z}(t) \mid \mathbf{z}_0,t_0]) +$$

$$\frac{1}{2} \sum_{k+l=3}^{\infty} \sum_{m+n=3}^{\infty} \frac{\lambda_{kl}^0 \lambda_{mn}^0}{k!\,l!\,m!\,n!} H_{k+m\,l+n}(\xi_1,\xi_2;\rho[\mathbf{Z}(t) \mid \mathbf{z}_0,t_0]) + \cdots \Bigg) \tag{7.128}$$

$$\lambda_{mn}^0 = \frac{\lambda_{mn}[\mathbf{Z}(t) \mid \mathbf{z}_0,t_0]}{\sigma[Y(t) \mid \mathbf{z}_0,t_0]^m\,\sigma[\dot{Y}(t) \mid \mathbf{z}_0,t_0]^n} \tag{7.129}$$

$$H_{mn}(\xi_1,\xi_2;\rho) = \frac{(-1)^{m+n}}{\varphi_2(\xi_1,\xi_2;\rho)} \frac{\partial^{m+n}}{\partial\xi_1^m \partial\xi_2^n} \varphi_2(\xi_1,\xi_2;\rho) \tag{7.130}$$

where $H_{mn}(\xi_1,\xi_2;\rho)$ denotes the bivariate Hermite polynomial, cf. (3.62). Eqs. (7.128), (7.129), (7.130) are also valid for the general non-linear system (1.86) under Gaussian white noise or compound Poisson process excitation, provided the joint cumulants of sufficiently high order can be calculated. In case of Gaussian responses one has $\lambda_{mn} = 0$, $m+n \geq 3$, and (7.128) reduces to (7.127).

It can be shown that $\lambda_{mn} \to \infty, m+n \geq 2$ as $\nu(t-t_0) \to 0$, implying a prohibitively slow convergence of the series expansion at the transition time intervals for which $\nu(t-t_0) \ll 1$. The indicated singularity can be circumvented by applying the expansion (7.128) to the first order for the transitional joint pdf in combination with a Gram-Charlier series for the conditioned transitional joint pdf $q_{\{\mathbf{Z}\}}^{(1)}(y,\dot{y},t \mid y_0,\dot{y}_0,t_0)$. Although this approach is numerically robust it has not been pursued in the following numerical example, which is based on (7.127) with all series expansions truncated up to and including the joint 6th order cumulants. It should be noted that truncation of the series expansion (7.128) at any finite order, corresponding to a finite order polynomial expansion of the log-characteristic function, cannot be mathematically justified. Actually, the theorem of Marcinkiewicz (1939) [7.14] states that either the log-characteristic function is a polynomial of the 2nd order corresponding to the Gaussian case, or joint cumulants of infinite order exist. Hence, the justification totally relies on the quality of the obtained results. Similar obstacles arise in the application of cumulant neglect closure schemes in moment methods of Markov systems as explained in section 3.2.

In order to compare the results for the compound Poisson excitation to those of the Gaussian white noise excitation the intensities of the impulses are assumed to be zero-mean normally distributed, $P \sim N(0,\sigma_P^2)$, with the variance σ_P^2 selected as follows

$$\nu\sigma_p^2 = 2\pi S_0 \tag{7.131}$$

The basis for this is the well-known convergence of the compound Poisson process to a Gaussian white noise as $\nu \to \infty$ under the restriction of (7.131).

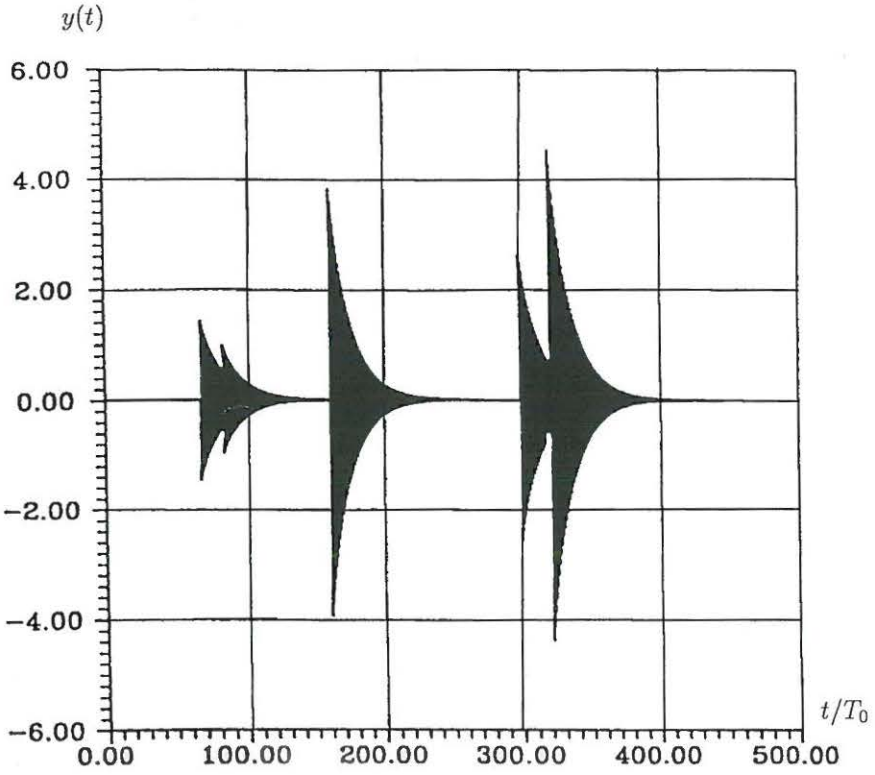

Fig. 7.14: Realization of displacement process for an SDOF system subjected to compound Poisson excitation with low mean arrival rate. $\zeta = 0.01$, $\nu T_0 = 0.01$, $P \sim N(0, \sigma_p^2)$.

The obtained numerical results have been shown as the dotted curves in fig. 7.13 for $\nu T_0 = 0.01$, 0.1, 1.0, 10.0. The convergence to Gaussian white noise may be considered to be attained for $\nu T_0 = 10.0$, so the noticed deviation from the full-line curve ($\nu T_0 = \infty$) can be attributed to the applied truncation of the Gram-Charlier series expansion.

It is remarkable that the height of the first stair level is almost constant for $\nu T_0 >\sim 0.1$, whereas the height is significantly smaller for $\nu T_0 = 0.01$. This suggests that the out-crossing events tend to become increasingly uncorrelated as $\nu T_0 \to 0$ under the restriction of (7.131). The explanation of this effect can be given with reference to the realizations of the response process in case of excitations with low mean arrival rate of impulses as shown in fig. 7.14. As seen the eigenvibrations have diminished substantially at the arrival of the next impulse.

Due to the stochastic independence of the pulse intensities, the eigenvibrations from consecutive impulses tend to become stochastically independent. Then, assume these eigenvibrations to be completely independent. The eigenvibrations $y(t) = h(t - t_1)P$, initiated by the last previous impulse P with arrival time t_1, will then cross out from the safe domain, if $h_{\max} \mid P \mid > b$. $h_{\max}$ is the maximum value of the impulse response function in (7.117). This provides the following asymptotic solution for the first-passage time probability density function

$$f_{T_1}(t \mid S_{t_0}) \simeq \frac{1}{\lambda_1} \exp(-\lambda_1 (t - t_0)) \tag{7.132}$$

$$\lambda_1 = P(h_{\max} \mid P \mid > b)\nu = \left(1 + F_p\left(-\frac{b}{h_{\max}}\right) - F_p\left(\frac{b}{h_{\max}}\right)\right)\nu \tag{7.133}$$

$$h_{\max} = \max_{t \in [t_0, \infty[} h(t) = h(t_{\max}) \tag{7.134}$$

$$t_{\max} = \frac{1}{\omega_0 \sqrt{1-\zeta^2}} \arctan\left(\frac{\sqrt{1-\zeta^2}}{b}\right) \tag{7.135}$$

where $F_P(p)$ signifies the probability distribution function of P. Since the system and the safe domain are time-invariant, and the state vector $\mathbf{Z}(t)$ is a Markov vector, the first-passage probability density function can be represented by the expansion (7.46). λ_1 as given by (7.133) is an approximation to the 1st eigenvalue of the forward and the backward integro-differential Chapman-Kolmogorov operators (the Kolmogorov-Feller operators) of the first-passage time problem. The approximation (7.132) has been plotted in fig. 7.14 as the lowest full-line curve. Since the indicated first-passage time curves for $\nu T_0 = 0.01$ are parallel, (7.133) turns out to be a good approximation to the limiting decay rate of the problem. In case of higher damping the approximation will be even better, because a faster decay of eigenvibrations then takes place, and the assumption of mutual stochastic independence of such eigenvibrations then is better fulfilled.

In case the diffusion vector $\mathbf{b}$ in (1.90) is a constant, the approximation (7.132) can be extended to the general non-linear system (1.86) exposed to compound Poisson process excitation. In this case the quantity $h_{\max}$, entering the expression (7.133) for the limiting decay rate, is alternatively defined by

$$h_{\max} = \max_{t \in [t_0, \infty[} c(t \mid 0, 1, t_0) \tag{7.136}$$

where $c(t \mid 0, 1, t_0)$ is the displacement eigenvibration $c(t \mid y_0, \dot{y}_0, t_0)$ with the initial value $\mathbf{z}_0^T = [y_0, \dot{y}_0] = [0, 1]$ at the time t_0.

In conclusion it has been demonstrated in example 7.7 that the integral equation (7.101) in combination with (7.103) may be used to obtain very accurate solutions for the first-passage time probability density function for simple linear systems exposed to Gaussian white noise. Approximate solutions can be obtained for similar non-linear systems exposed to white noise as well as for linear or non-linear systems exposed to compound Poisson process excitation.

7.4 Concluding remarks and comments

This chapter deals with the formulations and solutions of the first-passage problems. In the introduction to this chapter, the first passages, the 1st order out-crossing rate as well as the first-passage probability density and probability distribution functions are defined. Based on the statistics of the time interval spent in the safe domain, the formula for evaluating the first-passage probability density function is provided. The stochastic and deterministic start problems as well as the single and double barrier problems are formulated. Finally the extremes, i.e. the maximum and minimum value of the dynamic response process are defined (example 7.1). In the section 6.1 the Markov systems are considered. The reliability problem is first formulated based on the forward integro-differential Chapman-Kolmogorov equation, i.e. the boundary and initial value problem for the transition probability density function is formulated. Alternatively, based on the backward integro-differential Chapman-Kolmogorov equation the boundary and terminal value problems for the transition probability density function and for the first-passage time probability distribution function are stated. The equations for moments of the first-passage time are also given. The solution for the first-passage time probability distribution function in terms of eigenfunctions of the forward

and backward operators is provided. In example 7.2 the first-passage time probability distribution function for the Wiener (Brownian motion) process is evaluated. The section 6.2 covers the crossing theory. The nth order out-crossing rate is defined and the inclusion-exclusion series for the first-passage time probability distribution function is provided. The Rice formula for the mean out-crossings rate is also given. In a series of examples some specific problems are dealt with. The first one is that of evaluating the out-crossing frequency for the stationary Gaussian process (example 7.3). Next the expected number of local maxima per unit time is evaluated (example 7.4), by making use of the fact that a local maximum of the process is characterized by a down-crossing of the constant barrier by the velocity process. Next, out-crossing frequencies from time-invariant safe domains are determined (examples 6.4 and 6.5). The example 7.7 deals with the unconditional and conditional out-crossing rates for the Gaussian process. The integral equation governing the joint probability density of the first-passage time and of the associated out-crossing velocity is formulated in the case of one- and multidimensional process. The first-passage time probability density function can then be obtained as a marginal density function. Finally in the example 7.8 the solution for the single barrier stationary start first-passage time problem for single-degree-of-freedom oscillator subjected to a Gaussian white noise and to a Poisson driven train of impulses are given.

7.5 Bibliography

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CHAPTER 8

PATH INTEGRATION TECHNIQUES

The Navier-Stoke equation in two dimensions for non-compressible fluids can be reduced to the vorticity transport equation, which is an elliptic parabolic equation quite similar to the 2-dimensional Fokker-Planck equation. The numerical solution of this equation turns out to be difficult at large Reynolds numbers, where the convective term is dominating the viscosity term responsible for the diffusion of vorticity in the fluid. The fluid mechanicists have managed to come up with at least two stable numerical schemes to handle this problem. The one is termed the cloud-in-cell (or vortex-in-cell) technique, Baker [8.1], Leonard [8.2], and the other is the finite difference scheme with a socalled upwind differencing, Zienkiewicz and Taylor [8.3]. The very same techniques can be applied to the solution of the Fokker-Planck equation, where the drift vector components induce large convection terms in part of the solution space. Here, the cloud-in-cell technique is known as path integration, Wehner and Wolfer [8.4], Naess and Johnsen [8.10, 8.11], or cell-to-cell mapping Crandall, Chandiramani and Cook [8.7], Sun and Hsu [8.8, 8.9]. The basis of the methods is to reduce the time continuously and state continuous Markov vector process to a Markov chain by considering the process at discrete instants of time and by discretizing the sample space of the process. As shown below, path integration and cell-to-cell may be interpreted as merely different ways of evaluating the convolution integrals at transitions, and hence the discretization of the state space. In what follows the former name will be coined. Path integration has also been used for the solution of other parabolic equations in mathematical physics such as the Schrödinger equation in quantum mechanics, Kleinert [8.5], and the heat equation with convection. The present chapter 8 deals with the application of path integration methods for simple oscillatory systems exposed to Gaussian white noise and pulse processes. Chapter 9 deals with the solution of the same problems using the Petrov-Galerkin variational method. As shown in that chapter the Petrov-Galerkin method is identical to a finite difference method with upwind differencing on the difference operators for the convective terms.

The application of path integration in mechanics can be traced back to Crandall, Chandiramani and Cook [8.6]. This work was later taken up by Sun and Hsu [8.7, 8.8], who used a similar approach to the method. Naess and Johnsen [8.9, 8.10] and Johnsen [8.11] used an interpolation with B-splines between the nodal function values in order to enhance accuracy or reduce the number of unknowns. Further, a backstepping technique in time was developed to delimit the subset of the sample space, which will influence a certain state one time step ahead. The convolution integral is only performed over this subset, which enhances the speed of the calculations significantly. Inspired by the fact that the natural logarithms of quite many stationary probability densities is polynomial Moe [8.12] tried to achieve even better results upon performing the interpolation on the logarithm of the probability density. The first application of the path integration schemes to pulse driven systems was due to Köylüoğlu et al. [8.13, 8.14].

Let $q_{\{\mathbf{Z}\}}(\mathbf{z}, t|\mathbf{x}, t_0)$ signify the joint transition probability density function of the Markov vector process $\{\mathbf{Z}(t), t \in [t_0, t_1]\}$. Further, let $f_{\{\mathbf{Z}\}}(\mathbf{z}, t_i)$ be the 1st order probability density function at the time $t_i = t_0 + i\Delta t$, $i = 0, 1, 2, \ldots$. Next, the 1st order probability density function at the following instant of time t_{i+1} is given by the convolution integral

$$f_{\{\mathbf{Z}\}}(\mathbf{z}, t_{i+1}) = \int_{S_{t_i}} q_{\{\mathbf{Z}\}}(\mathbf{z}, t_{i+1}|\mathbf{x}, t_i) f_{\{\mathbf{Z}\}}(\mathbf{x}, t_i) d\mathbf{x} \tag{8.1}$$

From (8.1) the time continuous vector process has been discretized to the instants of time $t_i = t_0 + i\Delta t$. In order to discretize the state space, the sample space S_{t_i} is divided into a finite number N of small volumes $\Delta\mathbf{z}_k$ with an interior centre at $\mathbf{z}_k$. If $\Delta\mathbf{z}_k$ is sufficiently small for $q_{\{\mathbf{Z}\}}(\mathbf{z}_j, t_i + \Delta t|\mathbf{z}_k, t_i)$ and $f_{\{\mathbf{Z}\}}(\mathbf{z}_k, t_i)$ is approximately constant throughout the cell, the probability $\pi_j^{(i+1)}$ of being in the jth cell $\Delta\mathbf{z}_j$ at the time t_{i+1} is then given by the Riemann sum

$$\pi_j^{(i+1)} = \sum_{k=1}^{N} Q_{jk}^{(i)} \pi_k^{(i)} \quad , \quad j = 1, \ldots, N \tag{8.2}$$

$$\pi_k^{(i)} \simeq \Delta\mathbf{z}_k f_{\{\mathbf{Z}\}}(\mathbf{z}_k, t_i) \tag{8.3}$$

$$Q_{jk}^{(i)} \simeq \Delta\mathbf{z}_j q_{\{\mathbf{Z}\}}(\mathbf{z}_j, t_i + \Delta t|\mathbf{z}_k, t_i) \tag{8.4}$$

As seen from (8.4) the transition probability matrix relies on the transition probability density function. Since this is not known in general, nothing seems to be gained at first. However, the method benefits from the fact that asymptotic results for $q_{\{\mathbf{Z}\}}(\mathbf{z}_j, t_i + \Delta t|\mathbf{z}_k, t_i)$ can be derived for sufficiently small transition time intervals Δt. These solutions are known as *short time propagators*, and the various path integration methods are classified rather by its short time propagator rather than the way to evaluate the convolution numerically. (8.3), (8.4) are essentially the cell-to-cell mapping approach used by Sun and Hsu [8.7, 8.8]. More accurate solutions can be obtained, if the assumption of piecewise constant values of $f_{\{\mathbf{Z}\}}(\mathbf{z}, t)$ and $q_{\{\mathbf{Z}\}}(\mathbf{z}_j, t_i + \Delta t|\mathbf{z}_k, t_i)$ within a cell is replaced by a higher order interpolations between the adjacent nodal values of these quantities, such as the above-mentioned approach by Naess and Johnsen [8.9, 8.10] and Johnsen [8.11]. However, one and the same short time propagator may be used in both cases, so the use of different designations for these approaches does not seem well motivated. For this reason the designation cell-to-cell mapping will be abandoned in the present outline.

If the Markov process $\{\mathbf{Z}(t), t \in [t_0, t_1]\}$ is stationary and the intervals Δt between observations of the process are equidistant, the transition probability matrix becomes independent of the time of transition t_i, i.e. $\mathbf{Q}^{(i)} = \mathbf{Q}$, and the Markov chain becomes stationary, as well. The transition of states (8.4) can then be represented by the matrix equation

$$\boldsymbol{\pi}^{(i+1)} = \mathbf{Q}\boldsymbol{\pi}^{(i)} = \mathbf{Q}^i \boldsymbol{\pi}^{(0)} \tag{8.5}$$

where $\boldsymbol{\pi}^{(i)}$ is an N-dimensional vector of the state probabilities $\pi_k^{(i)}$ after i transitions, and $\boldsymbol{\pi}^{(0)}$ signifies the initial distribution at the time t_0.

In reliability problems absorbing states on the exit part of the boundary $\partial S_{t_i}^{(1)}$ are characterized by the transition probability $Q_{kk} = 1 \Rightarrow Q_{jk} = 0,\ j \neq k$. Each absorbing state then forms a *recurrent class*, whereas all non-absorbing states make up a single class of *transient states*, Osaki [8.17]. The transient states are all the cells within the safe domain S_{t_i}. The non-absorbed probability mass, which remains in the transient states after i transitions represents the reliability of the system at the time t_{i+1}.

In stochastic response problems no absorbing states within the sample space should be specified. However, in practical calculations one needs to delimit the sample space, so artificial absorbing states are specified at sufficiently distant positions for the probability of accessing these states to be small enough. The application of path integration methods to reliability problems and to stochastic response problems is then completely identical.

For the stochastic response problem in case of stationary Markov chains one may be interested in the question, whether a stationary distribution $\boldsymbol{\pi}^{(\infty)}$ to (8.5) exists, obtained after infinitely many transitions as $i \to \infty$, irrespective of the initial distribution $\boldsymbol{\pi}^{(0)}$ applied. Since the Markov chain in this case is *irreducible*, *positive recurrent* and *aperiodic*, the answer to this question is positive, see e.g. Osaki [8.17]. The limiting distribution must be invariant to further transitions, and can then be determined from the equation

$$\boldsymbol{\pi}^{(\infty)} = \mathbf{Q}\boldsymbol{\pi}^{(\infty)} \tag{8.6}$$

(8.6) determines $\boldsymbol{\pi}^{(\infty)}$ as the normalized eigenvector to a linear eigenvalue problem with the known eigenvalue $\lambda = 1$. This eigenvalue is simple, since the Markov chain is irreducible, and the solution $\boldsymbol{\pi}^{(\infty)}$ to (8.6) then is unique. Hence the coefficient matrix $\mathbf{I} - \mathbf{Q}$ has the rank $N - 1$ and (8.6) can be rearranged into a system of linear equations of the order $N - 1$. Alternatively, $\boldsymbol{\pi}^{(\infty)}$ can be obtained by iterating in the transition equation until convergence for an arbitrary initial value distribution $\boldsymbol{\pi}^{(0)}$ is attained. Since $\lambda = 1$ is the largest eigenvalue of $\mathbf{Q}$, this corresponds to the power method approach in numerical solution of linear eigenvalue problems. Since the eigenvalue $\lambda = 1$ is known, the former approach is normally the most efficient one. Moreover, for symmetric problems, such as double barrier problems, the size of eigenvector problem can be further reduced.

In sections 8.1 and 8.2 it is demonstrated how these results are obtained for systems driven by Wiener processes and for systems driven by generating source processes with jumps. In both cases the illustrative example is a non-linear, non-hysteretic SDOF oscillator. This is so, because at present path integration methods have not yet been extended to MDOF systems.

8.1 Path integration methods for Wiener process driven systems

The Wiener process driven system (2.72) is considered. At transitions from the state $\mathbf{Z}(t_i) = \mathbf{z}_k$ it is assumed that the transition time interval $t - t_i$ is sufficiently small for the drift vector $\mathbf{c}\big(\mathbf{Z}(t), t\big)$ to be approximated by a linear function of the state vector $\mathbf{Z}(t)$, and the diffusion matrix $\mathbf{d}\big(\mathbf{Z}(t), t\big)$ to be considered a constant as a function of $\mathbf{Z}(t)$, although it may still be a function of time. Hence, for $t \in]t_i, t_i + \Delta t]$ it is assumed

$$\mathbf{c}\big(\mathbf{Z}(t), t\big) \simeq \mathbf{A}(t) + \mathbf{B}(t)\mathbf{Z}(t) \tag{8.7}$$

$$\mathbf{d}\big(\mathbf{Z}(t), t\big) \simeq \mathbf{d}_0(t) \tag{8.8}$$

The process cannot perform any jumps, so (8.7) and (8.8) will certainly be acceptable, if only the transition time interval is sufficiently small. Since the approximated system is linear under Gaussian white noise excitation, the response becomes Gaussian too. This is the principle of *local Gaussianity*, valid for non-linear Markov systems, during small transition time intervals. The transition probability density function $q_{\{\mathbf{Z}\}}(\mathbf{z}, t|\mathbf{z}_k, t_i)$ then becomes

$$q_{\{\mathbf{Z}\}}(\mathbf{z}, t|\mathbf{z}_k, t_i) \simeq \frac{1}{(2\pi)^{\frac{n}{2}} \Big(\det\big(\boldsymbol{\kappa}(t)\big)\Big)^{\frac{1}{2}}} \exp\Big(-\frac{1}{2}\big(\mathbf{z} - \boldsymbol{\mu}(t)\big)^T \boldsymbol{\kappa}^{-1}(t)\big(\mathbf{z} - \boldsymbol{\mu}(t)\big)\Big) \tag{8.9}$$

where $\boldsymbol{\mu}(t) = E\big[\mathbf{Z}(t) \mid \mathbf{Z}(t_i) = \mathbf{z}_k\big]$ and $\boldsymbol{\kappa}(t) = E\big[\big(\mathbf{Z}(t) - \boldsymbol{\mu}(t)\big)\big(\mathbf{Z}(t) - \boldsymbol{\mu}(t)\big)^T \mid \mathbf{Z}(t_i) = \mathbf{z}_k\big]$ are the conditional mean value vector and the conditional covariance matrix function, respectively. These are the solutions of the following ordinary differential equations, (3.39), (3.46)

$$\frac{d}{dt}\boldsymbol{\mu}(t) = \mathbf{A}(t) + \mathbf{B}(t)\boldsymbol{\mu}(t) \quad , \quad t > t_i \quad , \quad \boldsymbol{\mu}(t_i) = \mathbf{z}_k \tag{8.10}$$

$$\frac{d}{dt}\boldsymbol{\kappa}(t) = \mathbf{B}(t)\boldsymbol{\kappa}(t) + \boldsymbol{\kappa}(t)\mathbf{B}^T(t) + \mathbf{d}_0(t)\mathbf{d}_0^T(t) \quad , \quad t > t_i \quad , \quad \boldsymbol{\kappa}(t_i) = \mathbf{0} \tag{8.11}$$

Normally, the linearization parameters $\mathbf{A}(t)$, $\mathbf{d}_0(t)$, $\mathbf{B}(t)$ depend on the initial state $\mathbf{z}_k$. In case of applications in path integration methods (8.10) and (8.11) should then be solved for initial values $\mathbf{z}_k$ in all N cells, and for each transition from t_i to t_{i+1}. In case of stationary excitation in time-invariant problems $\mathbf{A}(t), \mathbf{d}_0(t)$ and $\mathbf{B}(t)$ no longer depend explicitly on time.

Moreover, if the time interval Δt between transitions is constant, the Markov chain becomes stationary, and (8.10), (8.11) need only be integrated once for all N considered states.

The various path integration methods are characterized by the way the functions $\mathbf{A}(t)$, $\mathbf{d}_0(t)$ and $\mathbf{B}(t)$ are defined. In ascending order of complexity at least the following four specifications can be used

$$\left.\begin{aligned} \mathbf{A} &\equiv \mathbf{c}(\mathbf{z}_k, t_i) \\ \mathbf{d}_0 &\equiv \mathbf{d}(\mathbf{z}_k, t_i) \\ \mathbf{B} &\equiv \mathbf{0} \end{aligned}\right\} \tag{8.12}$$

$$\left.\begin{aligned} \mathbf{A}(t) &= \mathbf{c}(\mathbf{z}_k, t) - \mathbf{B}(t)\mathbf{z}_k \\ \mathbf{d}_0(t) &= \mathbf{d}(\mathbf{z}_k, t) \\ \mathbf{B}(t) &= \frac{\partial}{\partial \mathbf{z}^T}\mathbf{c}(\mathbf{z}_k, t) \end{aligned}\right\} \tag{8.13}$$

$$\left.\begin{aligned} \mathbf{A}(t) &= \mathbf{c}(\boldsymbol{\mu}(t), t) - \mathbf{B}(t)\boldsymbol{\mu}(t) \\ \mathbf{d}_0(t) &= \mathbf{d}(\boldsymbol{\mu}(t), t) \\ \mathbf{B}(t) &= \frac{\partial}{\partial \mathbf{z}^T}\mathbf{c}(\boldsymbol{\mu}(t), t) \end{aligned}\right\} \tag{8.14}$$

$$\left.\begin{aligned} \mathbf{A}(t) &= E\Big[\mathbf{c}\big(\mathbf{Z}(t), t\big)\Big] - \mathbf{B}(t)\boldsymbol{\mu}(t) \\ \mathbf{d}_0(t) &= E\Big[\mathbf{d}\big(\mathbf{Z}(t), t\big)\Big] \\ \mathbf{B}(t) &= E\Big[\frac{\partial}{\partial \mathbf{z}^T}\mathbf{c}\big(\mathbf{Z}(t), t\big)\Big] \end{aligned}\right\} \tag{8.15}$$

Insertion of (8.12) into (8.10), (8.11) provides the following solutions for the conditional moments

$$\left.\begin{aligned} \boldsymbol{\mu}(t) &\simeq \mathbf{z}_k + \mathbf{c}(\mathbf{z}_k, t_i)(t - t_i) \\ \boldsymbol{\kappa}(t) &\simeq \mathbf{d}(\mathbf{z}_k, t_i)\mathbf{d}^T(\mathbf{z}_k, t_i)(t - t_i) \end{aligned}\right\} \tag{8.16}$$

The short time propagator (8.9), (8.16) is identical to the asymptotic solution by Risken [8.18], which was used by Naess and Johnsen [8.9, 8.10]. Because of the crude level of approximation inherent in the linearization scheme (8.12) this approximation can only be expected to give accurate results for very small time steps. This may cause numerical inconveniences in the non-stationary case, where transitions from an initial distribution $\boldsymbol{\pi}^{(0)}$ are requested, and in reliability problems, because a large number of transitions (8.5) are requested for evolutions of the system in any finite time interval. However, if only a stationary distribution $\boldsymbol{\pi}^{(\infty)}$ is required, this problem is omitted using the eigenvalue approach (8.6) in favour of iterating (8.5) until convergence. The use of the method is due to the simple analytical results (8.16) for $\boldsymbol{\mu}(t)$ and $\boldsymbol{\kappa}(t)$.

Relationships (8.13) correspond to a first order Taylor expansion of $\mathbf{c}(\mathbf{Z}(t),t)$ and zero order Taylor expansion of $\mathbf{d}(\mathbf{Z}(t),t)$ from $\mathbf{Z}(t_i)=\mathbf{z}_k$. In this case the differential equations (8.10) and (8.11) become mutually uncoupled systems of differential equations, which usually have to be solved numerically. Because of the more accurate linearization somehow larger time steps Δt between transitions can be applied compared to the previous case.

In turn, relationship (8.14) corresponds to a first order Taylor expansion of $\mathbf{c}(\mathbf{Z}(t),t)$ and zero order Taylor expansion of $\mathbf{d}(\mathbf{Z}(t),t)$ from the running mean value $\boldsymbol{\mu}(t)$. Now, the systems of differential equations (8.10) and (8.11) become mutually coupled and non-linear. This does not imply any practical problems since these equations must be solved numerically anyway.

The unknown expectations at the right-hand sides of (8.15) are supposed to be evaluated by the running joint probability density function (8.9). Then the result corresponds to the equivalent linearization scheme in the expected least square sense by Atalik and Utku [8.16], cf. (10.26). The resulting expectations will be non-linear functions of $\boldsymbol{\mu}(t)$ and $\boldsymbol{\kappa}(t)$, resulting in mutually coupled systems of ordinary differential equations upon insertion into (8.10), (8.11). This method was first used by Sun and Hsu [8.8] (in their previous work Sun and Hsu [8.7] the transition probability matrix $\mathbf{Q}$ was obtained by a simulation procedure). The methods resulting from the assertions (8.14) and (8.15) are assumed to give results of the same quality. In both cases much larger transition time intervals Δt, than for the assertions (8.12) and (8.13), can be assumed. In some cases Δt can be chosen as large as $0.5T_0$, T_0 being the fundamantal eigenperiod of the linear oscillator.

Example 8.1: Duffing oscillator subjected to a Gaussian white noise

The Duffing oscillator exposed to Gaussian white noise is considered. Assuming the displacement and the velocity response has been normalized to one for the corresponding linear oscillator the drift vector and the diffusion vector becomes, cf. (3.70), (3.77)

$$\mathbf{c}(\mathbf{Z}(t),t)=\begin{bmatrix} Z_2 \\ -2\zeta Z_2 - Z_1 - \kappa Z_1^3 \end{bmatrix} \quad , \quad \mathbf{d}=\begin{bmatrix} 0 \\ 2\sqrt{\zeta} \end{bmatrix} \tag{8.17}$$

(8.17) will be analyzed with the parameter values $\zeta=0.03$, $\kappa=0.2$. Since $\mathbf{d}$ is constant in time and independent of the state vector no approximation is needed for this quantity. For the cases (8.18) - (8.21) the quantities $\mathbf{A}(t)$ and $\mathbf{B}(t)$ defining the equivalent linearization of the drift vector become

$$\left.\begin{aligned} \mathbf{A} &= \begin{bmatrix} z_{2,k} \\ -2z_{2,k} - z_{1,k} - \kappa z_{1,k}^3 \end{bmatrix} \\ \mathbf{B} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}\right\} \tag{8.18}$$

$$\left.\begin{aligned} \mathbf{A} &= \begin{bmatrix} 0 \\ 2\kappa z_{1,k}^3 \end{bmatrix} \\ \mathbf{B} &= \begin{bmatrix} 0 & 1 \\ -1 - 3\kappa z_{1,k}^2 & -2\zeta \end{bmatrix} \end{aligned}\right\} \tag{8.19}$$

$$\left.\begin{aligned}\mathbf{A}(t) &= \begin{bmatrix} 0 \\ 2\kappa\mu_1^3(t) \end{bmatrix} \\ \mathbf{B}(t) &= \begin{bmatrix} 0 & 1 \\ -1-3\kappa\mu_1^2(t) & -2\zeta \end{bmatrix}\end{aligned}\right\} \tag{8.20}$$

$$\left.\begin{aligned}\mathbf{A}(t) &= \begin{bmatrix} 0 \\ 2\kappa\mu_1^3(t) \end{bmatrix} \\ \mathbf{B}(t) &= \begin{bmatrix} 0 & 1 \\ -1-3\kappa(\mu_1^2(t)+\kappa_{11}(t)) & -2\zeta \end{bmatrix}\end{aligned}\right\} \tag{8.21}$$

Hence, the 1st order Taylor linearization from the running mean, (8.20), and the expected least-square minimization, (8.21), only deviate in the component B_{21} of the gradient matrix. (8.20) is significantly easier to a apply in case of hysteretic multi-degrees-of-freedom systems.

(8.20) will be applied for the determination of the stochastic response and reliability of the Duffing oscillator exposed to Gaussian white noise. In the numerical calculations the limits of the mesh are taken as $[-4,4]\times[-4,4]$, i.e. the mesh extends ± 4 stationary standard deviations of the linear oscillator. A uniform coarse 20×20 mesh is applied, so $\Delta x = 0.4$, $\Delta\dot{x} = 0.4$. The transition probability density at one transition time-interval has been obtained from (8.7), (8.8), (8.10), (8.11), (8.20) for all 400 cells using a 4th order Runge-Kutta scheme. The transition time-interval was selected as $\Delta t = \frac{T_0}{4}$, where $T_0 = 2\pi$ is the eigenperiod of the linear oscillator.

The results for the stationary marginal pdfs of the displacement and velocity have been shown in figs. 8.1 - 8.4. To emphasize on the tails of the distributions the results have also been shown in semi-logarithmic scale. The solid line curves represent the analytical solutions, whereas the results marked with • and ○ signify the numerical results obtained from iterating (8.5) $n = 60$ times and from the eigenvector solution (8.6), respectively. The iteration solution was started with deterministic start in the origin $\mathbf{Z}(0) = \mathbf{0}$. From these results it is concluded that the path integration method provides very accurate results for Wiener process driven systems at all levels of probability including the tails even with the applied coarse mesh and the applied large time step. In turn, the applicability of large time steps is a consequence of the relative accuracy of the linearization schemes (8.14) and (8.15). The calculation times were 39s for the iteration approach, and 139s for the eigenvector approach. Notice, for the present problem the number n of iteration required to achieve convergence is determined by the criterion $n\zeta\Delta t\omega_0 \simeq 2.8$.

Hence the benefit of using the iteration approach is partly due to the relatively large time step of $\Delta t = \frac{T_0}{4}$ and the large damping ratio of $\zeta = 0.03$. Johnsen [8.12] used a time step of $\Delta t = \frac{T_0}{2000\pi}$ and a 47×47 for the same problem with a damping ratio of $\zeta = 0.10$. In this case the eigenvector solution will turn out to be beneficial, especially in case of lower damping ratios.

In figs. 8.5 and 8.6 the time dependence of the first-passage time probability distribution function and the first-passage time probability density functions are shown for the same Duffing oscillator in case of a single constant barrier first-passage time problem with deterministic start at $\mathbf{Z}(0) = \mathbf{0}$, obtained by Monte-Carlo path integration and simulation, respectively. The path integration results are shown with an unbroken curve and the simulation results with a dotted curve. The barrier level is $2.0\sigma_X(\infty)$, where the stationary variance $\sigma_X^2(\infty) = 0.721023$ was obtained from the analytical 4th order cumulant neglect closure solution (3.86). The simulation results are based on 32000 independent realizations, obtained from numerical integration of the equations of motion (3.77) by means of the 4th order Runge-Kutta scheme with the time step $\Delta t = \frac{T_0}{40}$. Generation of realizations of a broad-band broken line zero mean Gaussian process was performed by the method of Penzien, Clough and Penzien [8.20]. As seen from fig. 8.5 the computed results overestimate the probability of failure somehow during the first period of the excitation, which provides a parallel translation of the first-pasage time probability distribution curve obtained by path integration compared to the one obtained by simulation. This effect can be attributed to the problem of convecting and diffusing the probability mass, initially concentrated at $\mathbf{Z}(0) = \mathbf{0}$, to the nodes of the mesh, with the relatively large transition time step of $T_0/4$ in combination with the applied coarse 20×20 mesh. The first-passage time probability density functions on fig. 8.6 were obtained by numerical differentiation, which caused the highly irregular behaviour.

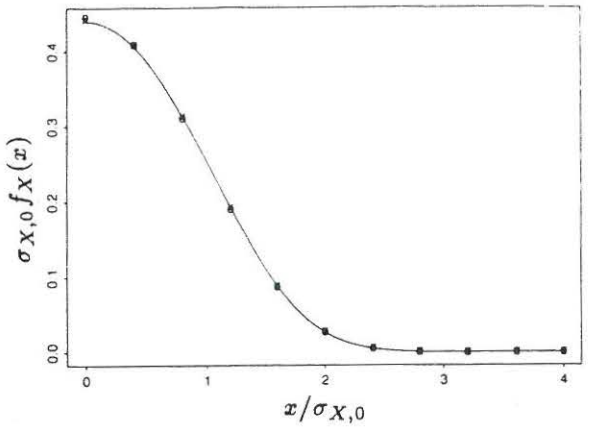

Fig. 8.1: Stationary pdf of displacement, $f_X(x)$. Linear scale. • iteration, ○ eigenvector solution.

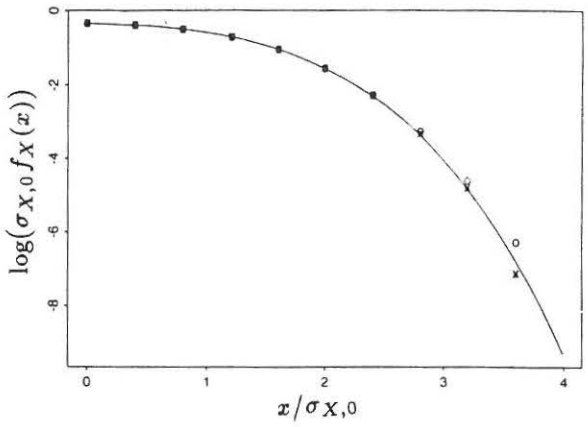

Fig. 8.2: Stationary pdf of displacement, $f_X(x)$. Semi-log scale. • iteration, ○ eigenvector solution.

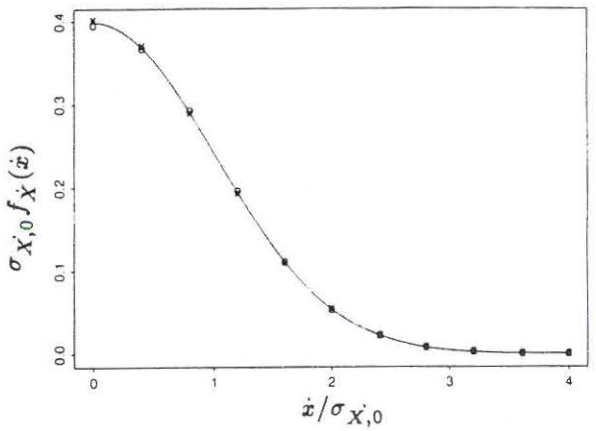

Fig. 8.3: Stationary pdf of velocity, $f_{\dot{X}}(\dot{x})$. Linear scale. • iteration, ○ eigenvector solution.

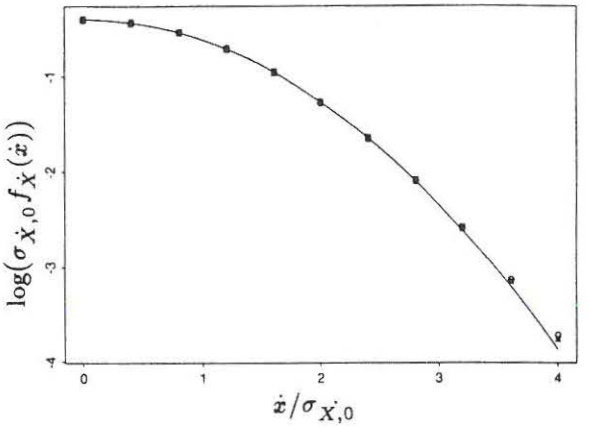

Fig. 8.4: Stationary pdf of velocity, $f_{\dot{X}}(\dot{x})$. Semi-log scale. • iteration, ○ eigenvector solution.

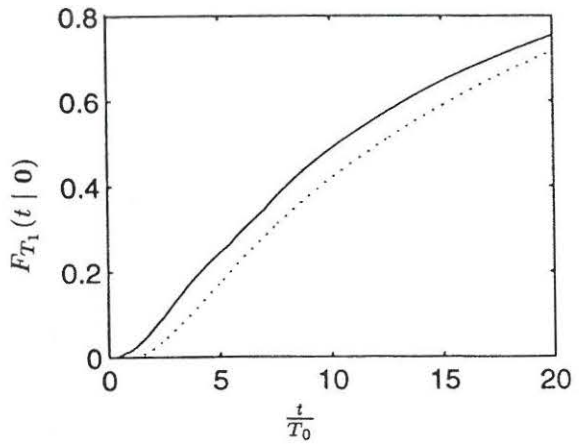

Fig. 8.5: First-passage time probability distribution function, $F_{T_1}(t \mid 0)$. Deterministic start with a single constant barrier, $b = 2.0\sigma_X(\infty)$. (—) Simulation, $(\cdots)$ path integration.

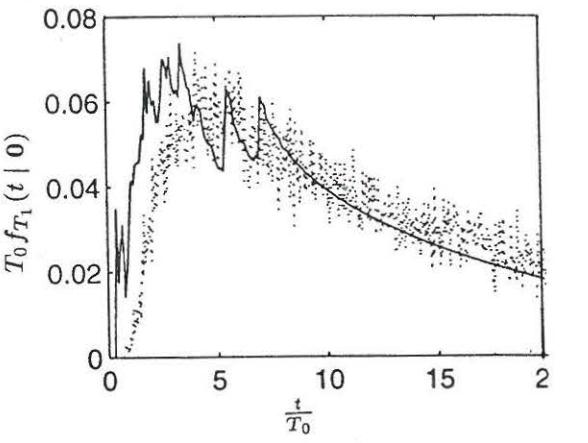

Fig. 8.6: First-passage time probability density function, $f_{T_1}(t \mid 0)$. Deterministic start with a single constant barrier, $b = 2.0\sigma_X(\infty)$. (—) Simulation, $(\cdots)$ path integration.

The general conclusion from this example is that path integration methods are useful both for stochastic response problems and for reliablity problems of SDOF non-linear oscillators. For stochastic response problems very coarse meshes may be used, whereas a somewhat finer mesh should be used in reliability problems. The main shortcoming of path integration methods is the rapid growth of the cpu time for problems of higher dimension than 2. According to Naess and Johnsen [8.10] the cpu time for the path integration method applied by them easily runs up to many hours on a work station (Dec 3100) for 3-dimensional problems. Hence supercomputing (parallel computing) becomes nescessary at even higher dimensions.

8.2 Path integration methods for systems driven by processes with jumps

In this section attention will be restricted to cases, where the system (1.105) is driven by a scalar generating source process with jumps $\{V(t), t \in [t_0, \infty[\}$.

Initially, the case of a system driven by a scalar compound Poisson process is considered. Let $q^{(n)}_{\{\mathbf{Z}\}}(\mathbf{z}, t|\mathbf{z}_k, t_i)$ be the transition probability density function of the state vector from the state $\mathbf{Z}(t_0) = \mathbf{z}_k$ on condition of exact n impulse arrivals in the time interval $[t_i, t[$, and let $P_{\{N\}}(n, t, t_i)$ denote the probability function of exactly n arrivals in this time interval as given by (1.22). Using the total probability theorem the unconditional transition probability density function can then be written as

$$q_{\{\mathbf{Z}\}}(\mathbf{z}, t|\mathbf{z}_k, t_i) = \sum_{n=0}^{\infty} P_{\{N\}}(n, t, t_i) q^{(n)}_{\{\mathbf{Z}\}}(\mathbf{z}, t|\mathbf{z}_k, t_i) \tag{8.22}$$

The 1st term in the sum (8.22) is identical to the 1st term in the modified closure (4.88). $q^{(0)}_{\{\mathbf{Z}\}}(\mathbf{z}, t|\mathbf{z}_k, t_i)$ describes the purely deterministic drift (eigenvibration) of the system from the state $\mathbf{Z}(t_i) = \mathbf{z}_k$, since the states are conditioned on no impulse arrival. Then $q^{(0)}_{\{\mathbf{Z}\}}(\mathbf{z}, t|\mathbf{z}_k, t_i)$ is given as

$$q^{(0)}_{\{\mathbf{Z}\}}(\mathbf{z}, t|\mathbf{z}_k, t_i) = \prod_{j=1}^{n} \delta\Big(z_j - e_j(t|\mathbf{z}_k, t_i)\Big) \tag{8.23}$$

where $\mathbf{e}(t|\mathbf{z}_k, t_i)$ with the components $e_i(t|\mathbf{z}_k, t_i)$ denotes the deterministic drift motion of the system from the initial state $\mathbf{z}_k$ at the time t_i. The vector $\mathbf{e}(t|\mathbf{z}_k, t_i)$ is the solution of the initial value problem originating from (1.105)

$$\frac{\partial}{\partial t}\mathbf{e}(t|\mathbf{z}_k, t_i) = \mathbf{c}\big(\mathbf{e}(t|\mathbf{z}_k, t_i), t\big) \quad , \quad t > t_i \quad , \quad \mathbf{e}(t_i|\mathbf{z}_k, t_i) = \mathbf{z}_k \tag{8.24}$$

For linear systems (8.24) may be solved analytically, depending on whether a fundamental basis of solutions can be found. This is the case of linear vibratory systems, for which modal decoupling can be used. For other linear systems and for non-linear systems (8.24) must be solved numerically.

The remaining conditional transition probability density functions, $q^{(n)}_{\{\mathbf{Z}\}}(\mathbf{z}, t|\mathbf{z}_k, t_i)$, $n \geq 1$, are all continuous functions without delta spikes, and of the same order of magnitude, cf. (4.88). Since $P_{\{N\}}(n, t, t_i) = \mathrm{O}\Big(\big(\nu(t_i)(t-t_i)\big)^n\Big)$, it follows that (8.22) can be written as

$$q_{\{\mathbf{Z}\}}(\mathbf{z}, t \mid \mathbf{z}_k, t_i) = P_0(t, t_i) q^{(0)}_{\{\mathbf{Z}\}}(\mathbf{z}, t|\mathbf{z}_k, t_i) +$$
$$\big(1 - P_0(t, t_0)\big) q^{(1)}_{\{\mathbf{Z}\}}(\mathbf{z}, t|\mathbf{z}_k, t_i) + \mathrm{O}\Big(\big(\nu(t_i)(t - t_i)\big)^2\Big) \tag{8.25}$$

where the probability of no impulse arrivals in the interval $[t_i, t[$, $P_0(t, t_i) = P_{\{N\}}(0, t, t_i)$, is given by (1.22). The asymptotic relationship (8.25) defines a class of short time propagators for path integration methods for Poisson driven systems. The specific formulation of (8.25) ensures that upon chopping the remainder, the quality of a genuine (actual) probability density function is preserved, i.e. the integral of the function over the sample space is exactly equal to one, for any choice of the transition time interval $t - t_i$. Further, it is important to notice that the remainder depends on the magnitude of the product $\nu(t_i)(t - t_i)$, rather than of the interval length $t - t_i$ itself. Hence, truncation is permitted if

$$\nu(t_i)(t - t_i) \ll 1 \tag{8.26}$$

For any finite transition interval (8.26) is more easily fulfilled for sparse pulse trains, for which $\nu(t_i)T_0 \ll 1$, T_0 being the fundamental eigenperiod of the corresponding linear structure, than for dense pulse trains. Hence, in contrast to the moment equations method of section 5.3 and the Petrov-Galerkin finite element formulation of chapter 9, which both work well for rather dense pulse trains and run into numerical instability for sparse pulses, the indicated path integration method is designed to work the other way around in providing the best results for sparse pulses.

On the other hand the coarseness of the mesh sets lower bounds for the admissibility of the transition time interval, $t - t_i$. This is so, because the distribution of the convected probability masses to the adjacent nodes in the mesh cannot be done sufficiently accurately, if $t - t_i$ is too small. Let $c^{(k)} = |\mathbf{c}(\mathbf{z}_k, t_i)|$ be the magnitude of the convection vector at the centre of the kth cell, and let $\Delta z^{(k)}$ be the diameter of the cell in the direction of $\mathbf{c}(\mathbf{z}_k, t_i)$. It is then required that a convection at least of the length $\Delta z^{(k)}$ should take place during the interval $t - t_i$ in all cells, i.e.

$$c^{(k)}(t - t_i) > \Delta z^{(k)} \quad \Rightarrow \quad C^{(k)} = \frac{c^{(k)}(t - t_i)}{\Delta z^{(k)}} > 1 \tag{8.27}$$

where $C^{(k)}$ is the *local Courant number* for the kth cell. Numerical instability and inaccurate results occur if $C^{(k)}$ becomes too small in some part of the mesh. For a SDOF system $c^{(k)} \simeq \Delta\dot{x}$ in the most critical cells close to origo. Then $C^{(k)} \simeq \frac{\Delta\dot{x}(t-t_i)}{\Delta x} \simeq \frac{\sigma_{\dot{X}}(t-t_i)}{\sigma_X} \simeq \omega_0(t - t_i)$ follows. Hence, the following lower bound for the transition time

interval is obtained from (8.27), $t - t_i > \frac{T_0}{2\pi}$. Therefore, a decrease in the transition time interval must be accompanied by a finer mesh to guarantee stability and higher accuracy.

At the determination of $q^{(1)}_{\{\mathbf{Z}\}}(\mathbf{z}, t|\mathbf{z}_k, t_i)$ the state is conditioned on exactly one impulse arrival in $[t_i, t[$. The arrival time (first-passage time), T_1, of this impulse has the probability density function, e.g. Osaki [8.17]

$$f_{T_1}(\tau) = \frac{\nu(\tau)}{\int_{t_i}^{t} \nu(u)du} \quad , \quad \tau \in [t_i, t[\tag{8.28}$$

Assume that the impulse arrives at the time $T_1 = \tau$ and has the strength $P = p$. Up to the time τ the system has been performing eigenvibrations from the initial state $\mathbf{z}_k$ at the time t_i. Then the state at the time τ^- is given by $\mathbf{Z}(\tau^-) = \mathbf{e}(\tau|\mathbf{z}_k, t_i)$. At the time τ a discontinuous change of the state of magnitude $\Delta\mathbf{Z}(\tau) = \mathbf{b}\big(\mathbf{e}(\tau|\mathbf{z}_k, t_i), \tau\big)p$ takes place, so the state at the time τ^+ becomes $\mathbf{Z}(\tau^+) = \mathbf{e}(\tau|\mathbf{z}_k, t_i) + \mathbf{b}\big(\mathbf{e}(\tau|\mathbf{z}_k, t_i), \tau\big)p$, cf. (1.105). $\mathbf{b}\big(\mathbf{Z}(\tau), \tau\big)$ signifies the diffusion vector of the system. Succeedingly, during the time interval $]\tau, t]$ the system continues performing eigenvibrations with these initial values, so the state at the time t is

$$Z_i(t) = e_i\Big(t\Big|\mathbf{e}(\tau|\mathbf{z}_k, t_i) + \mathbf{b}\big(\mathbf{e}(\tau|\mathbf{z}_k, t_i), \tau\big)p, \tau\Big) \tag{8.29}$$

Joint statistical moments conditioned on the state $\mathbf{z}_k$ at the time t_i and on exactly one impulse can then be evaluated from

$$E\big[Z_{i_1}(t)\cdots Z_{i_n}(t) \mid \mathbf{z}_k, t_i, N(t) = 1\big] \; =$$

$$\int_{t_i}^{t}\int_{-\infty}^{\infty}\prod_{j=1}^{n} e_{i_j}\Big(t\Big|\mathbf{e}(\tau|\mathbf{z}_k, t_i) + \mathbf{b}\big(\mathbf{e}(\tau|\mathbf{z}_k, t_i), \tau\big)p, \tau\Big) f_P(p) f_{T_1}(\tau) dp d\tau \tag{8.30}$$

At the derivation of (8.30), the mutual statistical independence of T_1 and P has been taken into account. In the same manner the joint characteristic function is obtained as

$$M_{\{\mathbf{Z}\}}\big(\boldsymbol{\theta}, t \mid \mathbf{z}_k, t_i, N(t) = 1\big) = E\big[\exp\big(i\boldsymbol{\theta}^T\mathbf{Z}(t)\big) \mid \mathbf{z}_k, t_i, N(t) = 1\big] =$$

$$\int_{t_i}^{t}\int_{-\infty}^{\infty}\exp\Big(i\sum_{j=1}^{n}\theta_j e_j\Big(t, \Big|\mathbf{e}(\tau|\mathbf{z}_k, t_i) + \mathbf{b}\big(\mathbf{e}(\tau|\mathbf{z}_k, t_i), \tau\big)p, \tau\Big)\Big) f_P(p) f_{T_1}(\tau) dp d\tau \tag{8.31}$$

Next, the function $q^{(1)}_{\{\mathbf{Z}\}}(\mathbf{z}, t|\mathbf{z}_k, t_i)$ can be obtained as an inverse Fourier transform of (8.31). Since this can be performed numerically since no severe singularities are present, the problem has, in principle, been solved. However, the indicated method

calls for an enormous numerical effort. Alternatively, $q^{(1)}_{\{\mathbf{Z}\}}(\mathbf{z}, t|\mathbf{z}_k, t_i)$ can be represented as a Gram-Charlier or Edgeworth expansion, see (3.63). Again, extensive computations are necessary, and only approximate solutions are obtained, due to the mandatory truncation of this expansion. Instead, two numerically stable methods, derived by Köylüoğlu, Nielsen and Iwankiewicz [8.13] and Köylüoğlu, Nielsen and Çakmak [8.14], respectively, are presented below, which requires much less numerical effort. The methods will be explained by a 2-dimensional state vector with the expanding domain $[-4\sigma_X(t), 4\sigma_X(t)] \times [-4\sigma_{\dot{X}}(t), 4\sigma_{\dot{X}}(t)]$ as shown in fig. 8.7, where $\sigma_X(t)$ and $\sigma_{\dot{X}}(t)$ signify the nonstationary standard deviations of the corresponding linear oscillator expressed to an equivalent Gaussian white noise. These can be analytically determined. For the sake of simplicity the method will be explained for the case where the diffusion vector $\mathbf{b}$ is assumed to be state- and time-independent. However the method is equally valid if this is not so.

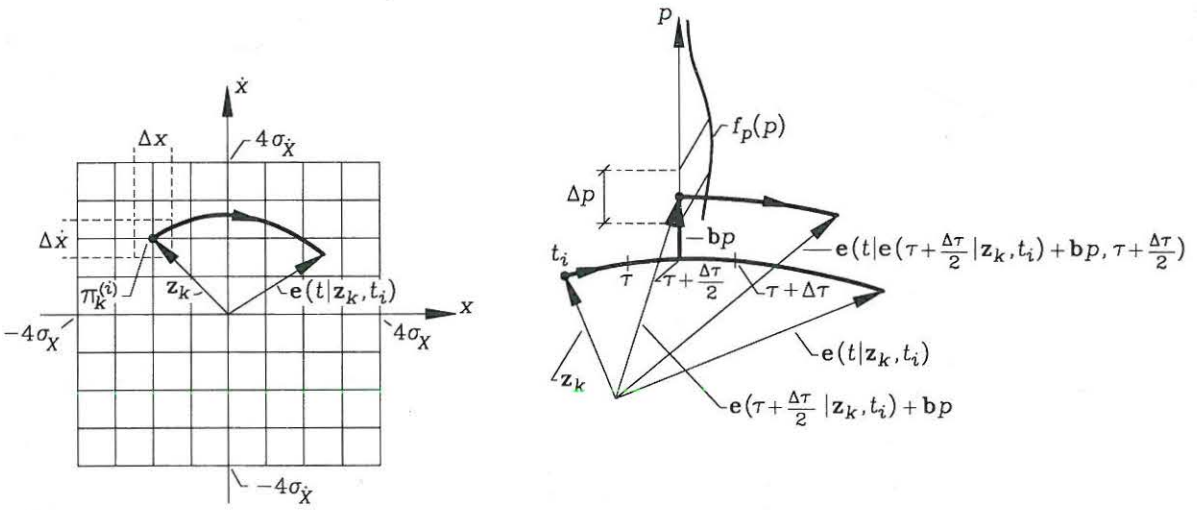

Fig. 8.7: Path integration of compound Poisson process driven system. Method 1: Discretization of phase plane, and convection and diffusion of probability mass.

On condition of being in node $\mathbf{z}_k$ at the time t_i, the transition probability Q_{jk} of being in node j at the time t can now be evaluated as shown in fig. 8.7. The discrete probability mass $\pi_k^{(i)}$ attached to node $\mathbf{z}_k$ at the time t_i is partly convected to $\mathbf{e}(t|\mathbf{z}_k, t_i)$ with the probability $P_0 = P_0(t, t_i)$ on condition that no impulses arrive in $]t_i, t]$. The rest of the probability mass $\pi_k^{(i)}(1 - P_0)$ is continuously distributed according to the transition probability density $q^{(1)}_{\{\mathbf{Z}\}}(\mathbf{z}, t|\mathbf{z}_k, t_i)$ of exactly one pulse arrival in $]t_i, t]$. Numerically, this is attained by dividing the time interval $[t_i, t[$ into a number of subintervals of the length $\Delta\tau$. In the subinterval $[\tau, \tau+\Delta\tau]$ a probability mass of magnitude $f_{T_1}(\tau)\Delta\tau\pi_k^{(i)}$ is lumped at the time $\tau + \frac{1}{2}\Delta\tau$, and succeedingly diffused and lumped along the direction $\mathbf{b}$ according to the probability density function $f_P(p)$, see fig. 8.7. At the position $\mathbf{e}(\tau + \frac{1}{2}\Delta\tau|\mathbf{z}_k, t_i) + \mathbf{b}p$ a probability mass of magnitude $f_{T_1}(\tau)\Delta\tau f_P(p)\Delta p(1 - P_0)\pi_k^{(i)}$ is

lumped. Next, this probability mass is convected to the position $\mathbf{e}\big(t|\mathbf{e}(\tau+\frac{1}{2}\Delta\tau|\mathbf{z}_k,t_i)+\mathbf{b}p,\tau+\frac{1}{2}\Delta\tau\big)$. Finally, at the time t all the convected probability masses are distributed to the 4 neighbouring nodal points, weighted according to their distances.

The second method is based on the following identity, valid for any $\tau\in]t_i,t[$

$$\mathbf{e}\big(t|\mathbf{e}(\tau|\mathbf{z}_k,t_i),\tau\big)=\mathbf{e}\big(t|\mathbf{z}_k,t_i\big) \tag{8.32}$$

The left- and right-hand sides of (8.32) just state that the oscillator arrives at the same position at the time t, if it starts on the very same trajectory at the position $\mathbf{e}\big(\tau|\mathbf{z}_k,t_i\big)$ at the time τ or at the position $\mathbf{e}\big(t_i|\mathbf{z}_k,t_i\big)=\mathbf{z}_k$ at the time t_i. From (8.32) the following Taylor expansion prevails for $\mathbf{Z}(t)$ given by (8.29)

$$\begin{aligned}\mathbf{Z}(t)=\mathbf{e}\big(t|\mathbf{e}(\tau|\mathbf{z}_k,t_i)+\mathbf{b}P,\tau\big)=\mathbf{e}\big(t|\mathbf{e}(\tau|\mathbf{z}_k,t_i),\tau\big)+\frac{\partial\mathbf{e}\big(t|\mathbf{e}(\tau|\mathbf{z}_k,t_i),\tau\big)}{\partial\mathbf{z}_k^T}\mathbf{b}P+\cdots=\\ \mathbf{e}\big(t|\mathbf{z}_k,t_i\big)+\mathbf{e}^{(1)}\big(t|\mathbf{z}_k,t_i\big)P+\mathbf{e}^{(2)}\big(t|\mathbf{z}_k,t_i\big)P^2+\cdots\end{aligned} \tag{8.33}$$

where

$$\left.\begin{aligned}&\mathbf{e}^{(1)}\big(t\big|\mathbf{z}_k,t_i\big)=\tfrac{\partial\mathbf{e}(t|\mathbf{z}_k,t_i)}{\partial\mathbf{z}_k^T}\mathbf{b}\\&\mathbf{e}^{(2)}\big(t\big|\mathbf{z}_k,t_i\big)=\tfrac{1}{2}\mathbf{b}^T\tfrac{\partial^2\mathbf{e}(t|\mathbf{z}_k,t_i)}{\partial\mathbf{z}_k\partial\mathbf{z}_k^T}\mathbf{b}\\&\vdots\end{aligned}\right\} \tag{8.34}$$

Equation (8.33) is basically a Taylor expansion in the impulse magnitude P. Notice that the random time τ entering the left-hand side of (8.33) has totally disappeared at the right-hand side. Instead the unknown Taylor expansion vectors $\mathbf{e}^{(1)},\mathbf{e}^{(2)},\ldots$ appear. Differential equations for these quantities can be derived upon multiple partial differentiations of (8.24) with respect to $\mathbf{z}_k$ and contractions with the vector $\mathbf{b}$

$$\left.\begin{aligned}&\tfrac{\partial}{\partial t}\big(\tfrac{\partial\mathbf{e}(t|\mathbf{z}_k,t_i)}{\partial\mathbf{z}_k^T}\big)=\tfrac{\partial\mathbf{c}\left(\mathbf{e}(t|\mathbf{z}_k,t_i),t\right)}{\partial\mathbf{z}^T}\tfrac{\partial\mathbf{e}(t|\mathbf{z}_k,t_i)}{\partial\mathbf{z}_k^T}\quad,\quad\tfrac{\partial\mathbf{e}(t_i|\mathbf{z}_k,t_i)}{\partial\mathbf{z}_k^T}=\mathbf{I}\ \Rightarrow\\&\tfrac{\partial}{\partial t}\mathbf{e}^{(1)}(t|\mathbf{z}_k,t_i)=\tfrac{\partial\mathbf{c}\left(\mathbf{e}(t|\mathbf{z}_k,t_i),t\right)}{\partial\mathbf{z}^T}\mathbf{e}^{(1)}(t|\mathbf{z}_k,t_i)\quad,\quad\mathbf{e}^{(1)}(t_i|\mathbf{z}_k,t_i)=\mathbf{b}\\&\tfrac{\partial}{\partial t}\mathbf{e}^{(2)}(t|\mathbf{z}_k,t_i)=\tfrac{\partial\mathbf{c}\left(\mathbf{e}(t|\mathbf{z}_k,t_i),t\right)}{\partial\mathbf{z}^T}\mathbf{e}^{(2)}(t|\mathbf{z}_k,t_i)+\\&\tfrac{1}{2}\mathbf{e}^{(1)T}(t|\mathbf{z}_k,t_i)\tfrac{\partial^2\mathbf{c}\left(\mathbf{e}(t|\mathbf{z}_k,t_i),t\right)}{\partial\mathbf{z}\partial\mathbf{z}^T}\mathbf{e}^{(1)}(t|\mathbf{z}_k,t_i)\quad,\quad\mathbf{e}^{(2)}(t_i|\mathbf{z}_k,t_i)=\mathbf{0}\\&\vdots\end{aligned}\right\} \tag{8.35}$$

where the partial differentiation with respect to the forward state $\mathbf{z}$ concerns the drift vector $\mathbf{c}(\mathbf{z},t)$. Relationships (8.24), (8.35) represent a coupled system of $n+n+n+\cdots$ non-linear 1st order differential equations for the determination of $\mathbf{e}(t|\mathbf{z}_k,t_i)$, $\mathbf{e}^{(1)}(t|\mathbf{z}_k,t_i)$, $\mathbf{e}^{(2)}(t|\mathbf{z}_k,t_i),\ldots$. The indicated method was given by Köylüoğlu, Nielsen and Çakmak [8.14] in a somewhat more involved outline. Rather than formulating the

differential equations for $\mathbf{e}^{(1)}(t|\mathbf{z}_k,t_i)$, $\mathbf{e}^{(2)}(t|\mathbf{z}_k,t_i),\ldots$ directly as shown in (8.35), differential equations were specified for the gradients $\frac{\partial}{\partial \mathbf{z}^T}\mathbf{e}(t|\mathbf{z}_k,t_i)$, $\frac{\partial^2}{\partial \mathbf{z}\partial \mathbf{z}^T}\mathbf{e}(t|\mathbf{z}_k,t_i),\ldots$. Taking the symmetry into consideration there will be $n+n^2+\frac{1}{2!}n^3+\cdots$ such different coupled differential equations. Hence, the specification (8.35) represents a significant saving if n is of even moderate magnitude. Even so the original formulation by Köylüoğlu, Nielsen and Çakmak [8.14] proved extremely advantageous compared to Monte Carlo simulation results as indicated below in example 8.2.

For a linear system the drift vector is a linear function of the state vector, so $\frac{\partial^2}{\partial \mathbf{z}\partial \mathbf{z}^T}\mathbf{c}(\mathbf{z},t) \equiv \mathbf{0}$. Due to the homogeneous initial values it then follows from (8.35) that $\mathbf{e}^{(2)}(t|\mathbf{z}_k,t_i) \equiv \mathbf{0}$. Generally, it can be shown that $\mathbf{e}^{(N)}(t_0|\mathbf{z}_k,t_0) \equiv \mathbf{0}$, $N \geq 1$, in this case. Hence, the Taylor expansion (8.33) becomes linear in P for linear systems. At small transition time intervals $\mathbf{c}(\mathbf{Z}(t),t)$ can be approximated by a linear function as stated by (8.7). Since this approach works well even for rather large time steps in case of white noise driven systems, it will also do so in the present case of compound Poisson driven systems. Consequently, it can be concluded that one can chop all terms higher than the 1st power in P in the expansion (8.33) with pretty good accuracy. Notice, that this approximation merely concerns the drift vector, and puts no restrictions on the magnitudes of the impulse strength P, as may appear at first sight.

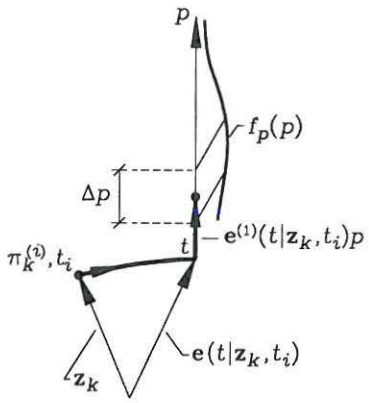

Fig. 8.8: Path integration of compound Poisson process driven system. Method 2: Convection and lumping of probability mass

As an approximation assuming the linear expansion in P in (8.33) even for non-linear systems, the convection and lumping of probability mass can be explained in the following way, see fig. 8.8. The probability mass $\pi_k^{(i)}$ at the node $\mathbf{z}_k$ at the time t_i is convected to the position $\mathbf{e}(t|\mathbf{z}_k,t_i)$ according to the 0th order term in (8.33), where a probability mass of magnitude $\pi_k^{(i)}P_0(t,t_i)$ is lumped. As specified by the 1st order term in (8.33) the remaining probability mass $\pi_k^{(i)}\big(1-P_0(t,t_i)\big)$ is distributed along the line with the direction $\mathbf{e}^{(1)}(t|\mathbf{z}_k,t_i)$ according to the probability density function $f_P(p)$ of the impulse intensity P. At the position $\mathbf{e}(t|\mathbf{z}_k,t_i)+\mathbf{e}^{(1)}(t|\mathbf{z}_k,t_i)p$ a probability mass of magnitude $\pi_k^{(i)}\big(1-P_0(t,t_i)\big)f_P(p)\Delta p$ is then lumped as sketched in fig. 8.8. Again, all lumped probability masses are redistributed to the 4 neighbouring nodal points, weighted according to their distances.

Example 8.2: Duffing oscillator subjected to compound Poisson process. Methods 1 and 2

The Duffing oscillator of example 8.1 is considered again, when driven by a stationary compound Poisson process.

For Method 1 the following system data are used $\zeta = 0.01$, $\kappa = 0.2$. The impulse strength of the compound Poisson process is assumed to be zero-mean normally distributed, $P \sim N(0, \sigma_P^2)$ with the variance chosen, so $\frac{\nu\sigma_P^2}{4\zeta} = 1$, corresponding to the stationary standard deviations $\sigma_X(\infty) = \sigma_{\dot{X}}(\infty) = 1$ of a linear oscillator exposed to an equivalent Gaussian white noise. Three values of ν are considered, namely $\nu = 0.01\omega_0 = \frac{0.02\pi}{T_0}$, $\nu = 0.1\omega_0 = \frac{0.2\pi}{T_0}$, $\nu = 1.0\omega_0 = \frac{2\pi}{T_0}$, which may be categorized as the cases of sparse, medium level and dense pulse arrival rates, respectively. Basically, the path integration has been performed by a uniform 20 × 20 mesh with the limits $[-4, 4] \times [-4, 4]$. However, in the case of $\nu = 0.01\omega_0 = \frac{0.02\pi}{T_0}$, where very peaked distributions occur at the origin, a non-uniform 25 × 25 mesh has been applied with a 4 times finer spacing close to origin. In establishing the transition probability matrix, all convection problems have been solved numerically, using a 4th order Runge-Kutta scheme. The transition time interval Δt is passed by 3 uniformly distributed steps for diffusion, i.e. $\Delta\tau = \frac{1}{3}\Delta t$, see fig. 8.7. An ensemble of 100000 Monte Carlo samples of the state vector is generated in order to estimate the stationary joint probability density of $X(t)$ and $\dot{X}(t)$. The system is performing eigenvibrations between impulse arrivals, which are determined upon numerical integration of (8.24). Assume, that an impulse of the strength p arrives at the time τ, where the state is $\mathbf{z}(\tau^-) = \mathbf{e}(\tau|\mathbf{z}_k, t_i)$. After the impulse the state is changed to $\mathbf{z}(\tau^+) = \mathbf{e}(\tau|\mathbf{z}_k, t_i) + \mathbf{b}(\mathbf{e}(\tau|\mathbf{z}_k, t_i), \tau)p$. Eigenvibrations in the succeeding interval is then calculated with the indicated initial values $\mathbf{z}(\tau^+)$. Samples of the interarrival times between pulse arrivals are generated from the knowledge that they are independent and identically distributed with the parameter ν. Samples p of the normal distributed pulse strength P can be generated independently of the interarrival times from the assumptions of the compund process, see section 1.1.2. Stationarity of the response is assumed after $50T_0$, at which time the stationary distributions have been sampled. In all cases, simulation results for the pdfs have been obtained with the same class-width as applied in the path integration scheme.

In figs. 8.9 - 8.12 are shown the stationary marginal probability density functions of the displacement and the velocity for the case of sparse pulse arrivals $\nu = 0.01\omega_0$. To emphasize on the tails the results have been indicated both in linear and semi-logarithmic scale. Transition is passed with the relatively large transition time interval $\Delta t = T_0$, so $\nu\Delta t = 0.02\pi$. The stationary joint probability density functions have been obtained from (8.5) with start in the origin using 100 iterations until stationarity. As seen the probability density functions reveal pronounced peaks at the origin. This is because the vibrations for sparse pulse arrivals almost extinct between the impulses as it can be observed in fig. 6.14. The oscillator is then in a position close to the origin for a large part of time, and hence there is a high probability density for the oscillator to be there. Since the dissipation of impulses is increased at large values of ζ, the peaks will even increase as the damping ratio is increased. Actually, the controlling parameter for the peakedness turns out to be the fraction $\frac{\zeta\omega_0}{\nu}$. In order to catch the peaks, results for the finer 25 × 25 mesh have also been obtained as explained above. The stationary standard deviations obtained from path integration are $\sigma_X(\infty) = 0.787$ and $\sigma_{\dot{X}}(\infty) = 0.958$, respectively. The corresponding Monte Carlo simulation results are $\sigma_X(\infty) = 0.799$ and $\sigma_{\dot{X}}(\infty) = 0.999$. Especially for the fine mesh high accuracy is achieved, when compared to Monte Carlo simulations.

Figs. 8.13 - 8.16 show the corresponding results for the case of medium level pulse arrivals $\nu = 0.1\omega_0$. The transition time interval is still kept at $\Delta t = T_0$, so $\nu\Delta t = 0.2\pi$. The stationary standard deviations obtained from path integration are $\sigma_X(\infty) = 0.868$ and $\sigma_{\dot{X}}(\infty) = 1.022$, respectively, whereas the corresponding Monte Carlo simulation results are $\sigma_X(\infty) = 0.842$ and $\sigma_{\dot{X}}(\infty) = 0.999$. As seen from the results high accuracy compared to Monte Carlo simulation is obtained even in this case with $\nu\Delta t = 0.2\pi$, which may be considered the outermost limit for the theory.

In figs. 8.17 - 8.20 the results for the case of dense pulse arrivals $\nu = 1.0\omega_0$ are shown. Under the restriction that $\nu E[P^2]$ is kept constant at the value $4\zeta\omega_0^3 m^2$ the response then resembles that of a

Gaussian white noise driven system. For this reason the analytical solution for the white noise case has also been shown. Stationarity was achieved after 50 iterations of (8.5). Even though the transition time was reduced to $\frac{T_0}{2}$ the method is now performing rather poor. Of course this is due to fact that the restriction on the method, (8.26), is no longer fulfilled with the present value of $\nu\Delta t = \pi$. The exact standard deviations for the white noise case are $\sigma_X(\infty) = 0.851$ and $\sigma_{\dot{X}}(\infty) = 1.000$. The corresponding results for Monte Carlo simulation are $\sigma_X(\infty) = 0.852$ and $\sigma_{\dot{X}}(\infty) = 0.997$. The conclusion is that for dense pulse arrivals an equivalent white noise approximation is doing better than the indicated path integration method.

Figs. 3-21 and 3-22 show the time dependence of the reliability function $R(t|\mathbf{0}) = 1 - F_{T_1}(t|\mathbf{0})$ and the first-passage time probability density function $f_{T_1}(t|\mathbf{0})$ for the case of sparse pulse arrivals $\nu = 0.01\omega_0$. The considered first-passage problem has symmetrically constant double barriers, $b = -a = 1.0\sigma_{Z_1,0}$ with deterministic start problem in $\mathbf{Z}(0) = \mathbf{0}$. Results obtained by path integration with a transition time interval $\Delta t = \frac{T_0}{2}$ have been compared to those obtained by Monte Carlo simulations. The simulation results are based on 100000 independent sample functions obtained by numerical integration with the average time step $\frac{T_0}{40}$, by means of a 4th order Runge-Kutta scheme. The first-passage time probability density functions for path integration and Monte Carlo simulation have been obtained by numerical differentiation of the reliability function. As is the case for the white noise case, cf. figs. 8.5 - 8.6, the path integration method underestimates the probability of failure during the first periods of excitation, because of the large transition time interval, made necessary by the coarseness of the applied uniform 20×20 mesh.

Figs. 8.21 and 8.22 show the time dependence of the reliability function $R(t|\mathbf{0}) = 1 - F_{T_1}(t|\mathbf{0})$ and the first-passage time probabilty density function $f_{T_1}(t|\mathbf{0})$ for the case of sparse pulse arrivals $\nu = 0.01\omega_0$. The considered first-passage problem has symmetrically constant double barriers, $b = -a = 1.0\sigma_{Z_1,0}$ with deterministic start problem in $\mathbf{Z}(0) = \mathbf{0}$. Results obtained by path integration with a transition time interval $\Delta t = \frac{T_0}{2}$ have been compared to those obtained by Monte Carlo simulations. The simulation results are based on 100000 independent sample functions obtained by numerical integration with the average time step $\frac{T_0}{40}$, by means of a 4th order Runge-Kutta scheme. The first-passage time probability density functions for path integration and Monte Carlo simulation have been obtained by numerical differentiation of the reliability function. As is the case for the white noise case, cf. figs. 8.5 - 8.6, the path integration method underestimates the probability of failure during the first periods of excitation, because of the large transition time interval, made necessary by the coarseness of the applied uniform 20×20 mesh.

Figs. 8.23 and 8.24 show the corresponding results for the case of medium level pulse arrivals, $\nu = 0.1\omega_0$. The obtained results have also been compared to those obtained by approximating the excitation process with an equivalent Gaussian white noise excitation, which was obtained upon solving the boundary value problem (6.37), using a Petrov-Galerkin variational formulation as explained in Chapter 9. As seen the results for $\nu = 0.1\omega_0$ deviate significantly from those for the white noise case, whereas the path integration results are in much better agreement with Monte Carlo simulation.

Figs. 8.25 - 8.26 show the reliability function and first-passage time probability density function for a single constant barrier problem problem with stationary start in the safe domain. The barrier level is $b = 1.0\sigma_{X,0}$. The initial distribution $\boldsymbol{\pi}^{(0)}$ was obtained from the analytical solution for the Gaussian white noise case, which was lumped at the nodes of the mesh, and normalized, so the probability mass within the safe domain amounts to 1. The simulation results were obtained by ergodic sampling based on (6.16) with 10000 out-crossing events. As seen, the staircase character of the first-passage time probability density function during the initial stages of first-passages cannot be caught by the path integration results. Nevertheless, the correct limiting exponential decay, corresponding to a discrete eigen-spectrum of the backward Kolmogorov-Feller operator with absorbing exit boundaries, cf. (6.64), is clearly captured by the path integration method.

Figs. 8.27 and 8.28 show the corresponding solutions for the case of medium level arrival rate of impulses, $\nu = 0.1\omega_0$. Notice that the transition time interval has been reduced to $\Delta t = \frac{T_0}{2}$. Again the correct limiting decay rate is captured.

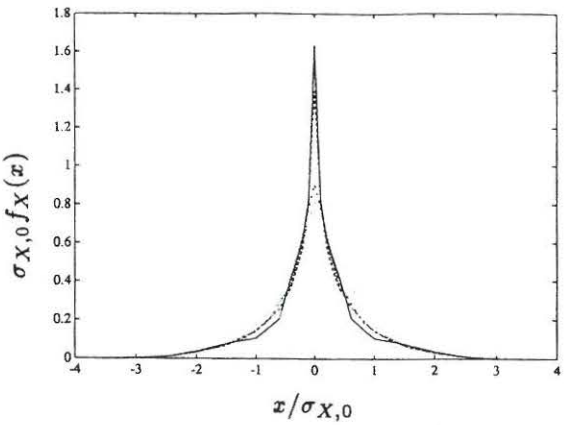

Fig. 8.9: Method 1. Pdf of displacement, $f_X(x)$. Linear scale. $\nu = 0.01\omega_0$, $\Delta t = T_0$. Fine mesh: (- - -) simulation, (—) path integration. Uniform mesh: (-·-·-) simulation, (· · ·) path integration.

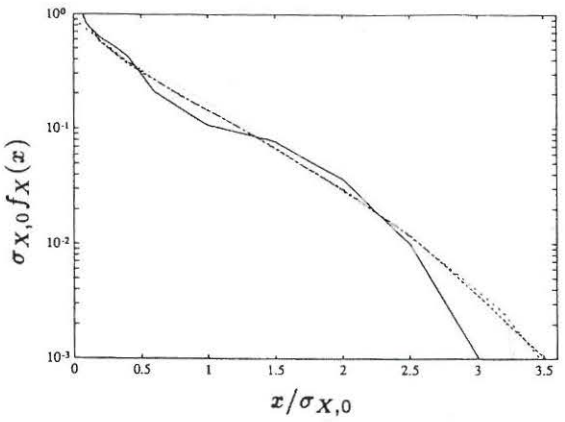

Fig. 8.10: Method 1. Pdf of displacement, $f_X(x)$. Semi-log scale. $\nu = 0.01\omega_0$, $\Delta t = T_0$. Fine mesh: (- - -) simulation, (—) path integration. Uniform mesh: (-·-·-) simulation, (· · ·) path integration.

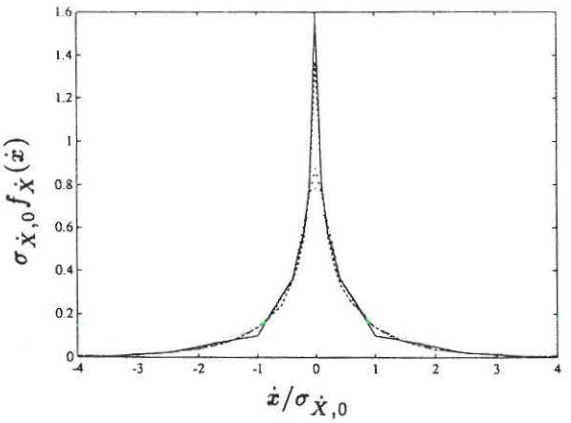

Fig. 8.11: Method 1. Pdf of velocity, $f_{\dot{X}}(\dot{x})$. Linear scale. $\nu = 0.01\omega_0$, $\Delta t = T_0$. Fine mesh: (- - -) simulation, (—) path integration. Uniform mesh: (-·-·-) simulation, (· · ·) path integration.

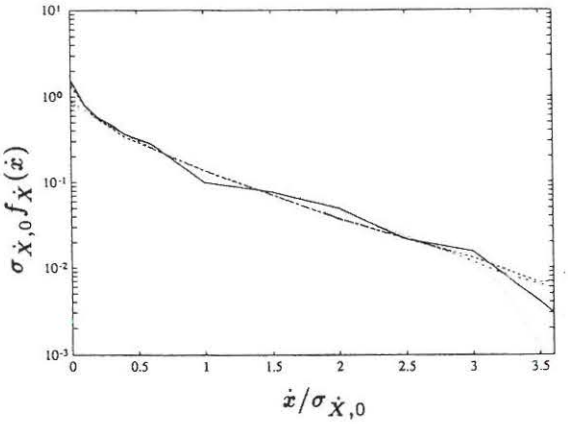

Fig. 8.12: Method 1. Pdf of velocity, $f_{\dot{X}}(\dot{x})$. Semi-log scale. $\nu = 0.01\omega_0$, $\Delta t = T_0$. Fine mesh: (- - -) simulation, (—) path integration. Uniform mesh: (-·-·-) simulation, (· · ·) path integration.

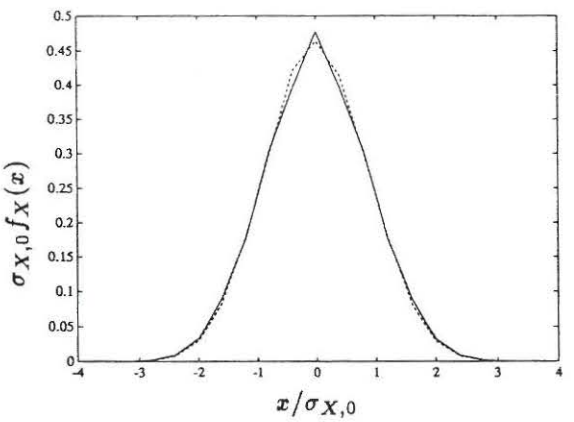

Fig. 8.13: Method 1. Pdf of displacement, $f_X(x)$. Linear scale. $\nu = 0.1\omega_0$, $\Delta t = T_0$. Uniform mesh: (- - -) simulation, (—) path integration.

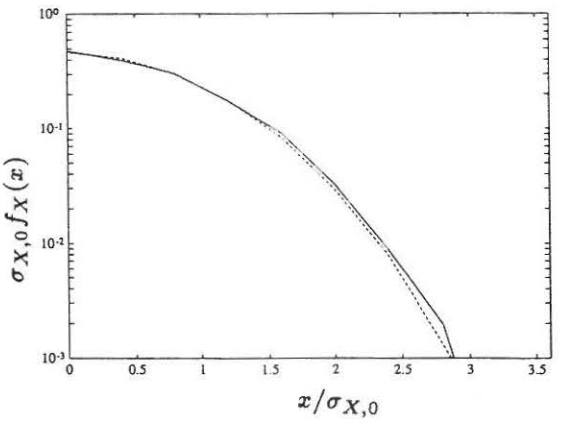

Fig. 8-14: Method 1. Pdf of displacement, $f_X(x)$. Semi-log scale. $\nu = 0.1\omega_0$, $\Delta t = T_0$. Uniform mesh: (- - -) simulation, (—) path integration.

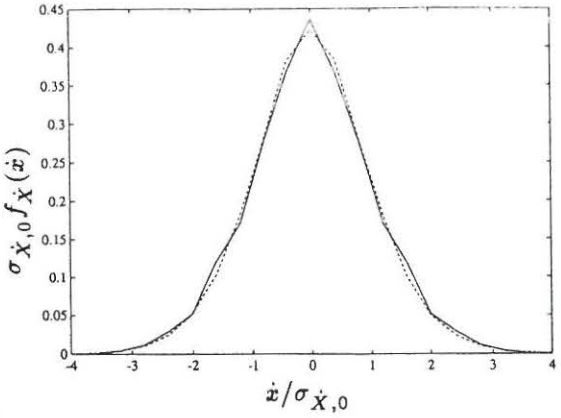

Fig. 8.15: Method 1. Pdf of velocity, $f_{\dot{X}}(\dot{x})$. Linear scale. $\nu = 0.1\omega_0$, $\Delta t = T_0$. Uniform mesh: (- - -) simulation, (—) path integration.

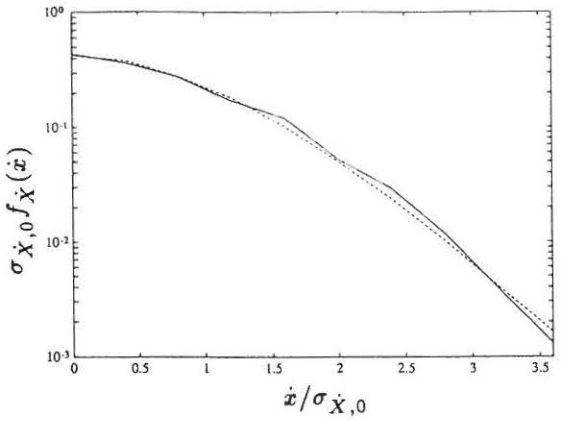

Fig. 8.16: Method 1. Pdf of velocity, $f_{\dot{X}}(\dot{x})$. Semi-log scale. $\nu = 0.1\omega_0$, $\Delta t = T_0$. Uniform mesh: (- - -) simulation, (—) path integration.

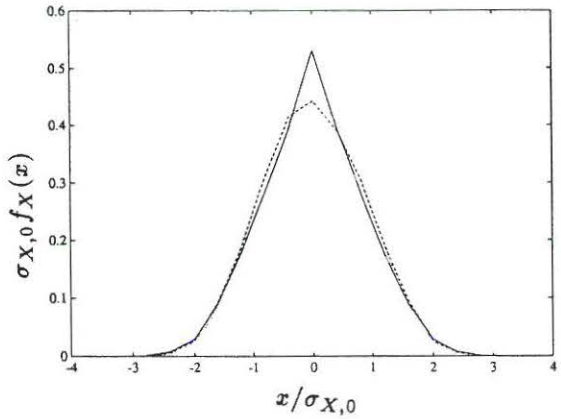

Fig. 8.17: Method 1. Pdf of displacement, $f_X(x)$. Linear scale. $\nu = 1.0\omega_0$, $\Delta t = \frac{T_0}{2}$. Uniform mesh: (- - -) simulation, (—) path integration, (· · ·) exact solution to white noise excitation.

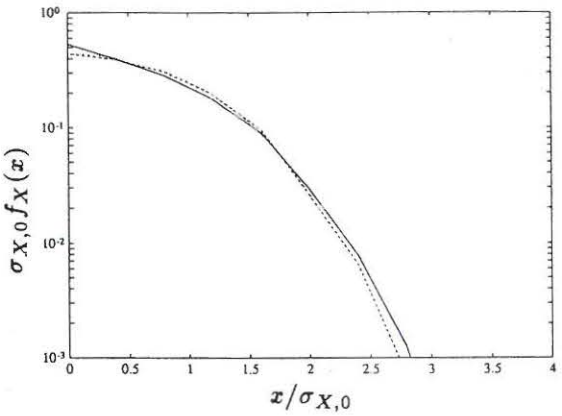

Fig. 8.18: Method 1. Pdf of displacement, $f_X(x)$. Semi-log scale. $\nu = 1.0\omega_0$, $\Delta t = \frac{T_0}{2}$. Uniform mesh: (- - -) simulation, (—) path integration, (· · ·) exact solution to white noise excitation.

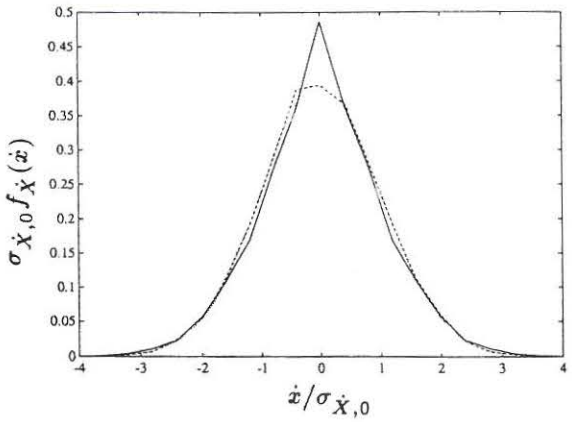

Fig. 8.19: Method 1. Pdf of velocity, $f_{\dot{X}}(\dot{x})$. Linear scale. $\nu = 1.0\omega_0$, $\Delta t = \frac{T_0}{2}$. Uniform mesh: (- - -) simulation, (—) path integration, (· · ·) exact solution to white noise excitation.

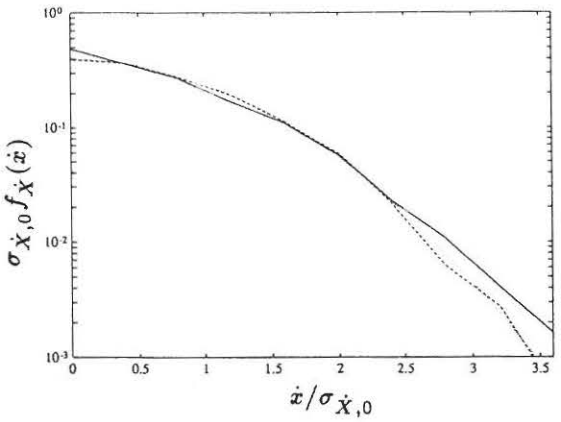

Fig. 8.20: Method 1. Pdf of velocity, $f_{\dot{X}}(\dot{x})$. Semi-log scale. $\nu = 1.0\omega_0$, $\Delta t = \frac{T_0}{2}$. Uniform mesh: (- - -) simulation, (—) path integration, (· · ·) exact solution to white noise excitation.

For Method 2 a slightly larger damping ratio of $\zeta = 0.03$ has been used, corresponding in the structural system used in example 8.1. The impulse strength of the compound Poisson process is still assumed to be zero-mean normally distributed, $P \sim N(0, \sigma_P^2)$, with the variance chosen, so $\frac{\nu\sigma_P^2}{4\zeta} = 1$. The mean arrival rates of impulses are chosen as $\nu = \frac{0.1}{\pi}\omega_0$. Then the fraction $\frac{\zeta\omega_0}{\nu}$ is approximately the same as for the system in figs. 8.9 - 8.12, and markedly peaked behaviour should be expected. In establishing the transitional probability matrix, the coupled differential equations (8.32), (8.35) have been solved by a 4th order Runge-Kutta scheme. The path integration is performed by a uniform 20×20 mesh with the limits $[-4, 4] \times [-4, 4]$, and the transition time interval is taken as $\Delta t = T_0$. Solutions for the stationary marginal pdfs of $X(\infty)$ and $\dot{X}(\infty)$ have been obtained both by iteration of (8.5) with start in the origin using 50 iterations until stationarity, and by the eigenvector solution (8.6). Monte Carlo simulation results have been based on an ensemble of 100000 Monte Carlo realizations of $X(t)$ and $\dot{X}(t)$. Stationarity of the response is assumed after $50T_0$, at which time the stationary distributions have been sampled, using the same class-width as applied in the path integration scheme.

In figs. 8.27 - 8.32 the obtained stationary marginal probability density functions of the displacement and the velocity in linear and semi-logarithmic scale are shown. Upon comparison with the corresponding results in figs. 8.9 - 8.12 obtained by Method 1 with a uniform mesh it is concluded that the present distribution and lumping scheme is performing at least equally well. At the same time the scheme is significantly simpler and faster to use. The necessary computing times for the path integration method based on iteration, on eigenvector solution and for the Monte Carlo simulation were 37s, 31s, and 13950s, respectively, which concludes that path integration offers extreme computational advantages over the Monte Carlo simulation method.

The main conclusion to be drawn from this example is that both the convection and diffusion schemes explained in figs. 8.7 and 8.8 provide accurate estimates of the probability density functions, even with the mesh as coarse as 20×20. In reliability applications, the probabilty of failure is underestimated during the initial periods of first-passages with such a mesh, and a finer mesh should be applied. However, even with the coarse mesh the methods are capable of capturing the correct limiting exponential decay of the reliability function and the first-passage time probability density functions.

The path integration schemes of Method 1 and Method 2 are both based on the asymptotic expansion (8.25), which is valid under the restriction (8.26). For the present example of a Duffing oscillator with moderate non-linearity parameter this criterion turns out to be fulfilled for $\nu \leq 0.1\omega_0$. As mentioned in Chapter 5, moment methods work at best at the other extreme of very dense pulse arrivals. Upon using the modified cumulant neglect closure schemes (4.99), (4.100), devised for closure at the order $N = 4$, it was possible to extend the applicability of moment methods for the present example to mean arrival rates down to $\nu \geq 0.05\omega_0$. Hence the whole range of mean arrival rates has been covered by the two methods in combination.

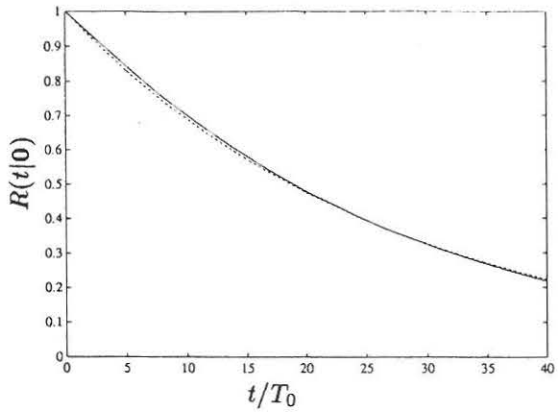

Fig. 8.21: Method 1. Reliability function, $R(t|\mathbf{0})$. Deterministic start, double barrier problem. $b = -a = \sigma_{Z_1,0}$, $\nu = 0.01\omega_0$, $\Delta t = \frac{T_0}{2}$. Uniform mesh: (- - -) simulation, (—) path integration.

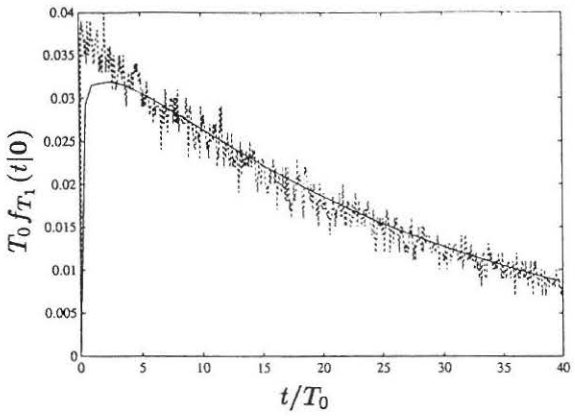

Fig. 8.22: Method 1. First-passage time pdf, $f_{T_1}(t|\mathbf{0})$. Deterministic start, double barrier problem. $b = -a = \sigma_{Z_1,0}$, $\nu = 0.01\omega_0$, $\Delta t = \frac{T_0}{2}$. Uniform mesh: (- - -) simulation, (—) path integration.

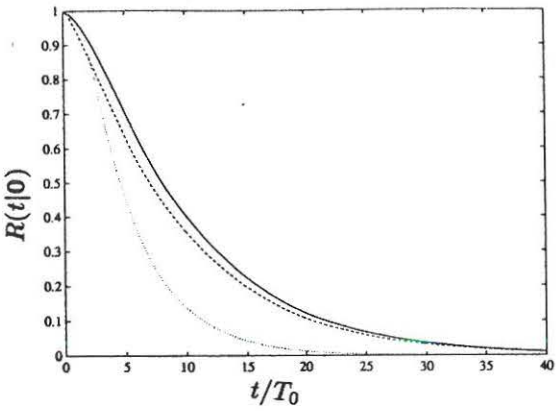

Fig. 8.23: Method 1. Reliability function, $R(t|\mathbf{0})$. Deterministic start, double barrier problem. $b = -a = \sigma_{Z_1,0}$, $\nu = 0.1\omega_0$, $\Delta t = \frac{T_0}{2}$. Uniform mesh: (- - -) simulation, (—) path integration, mesh: (· · ·) white noise excitation.

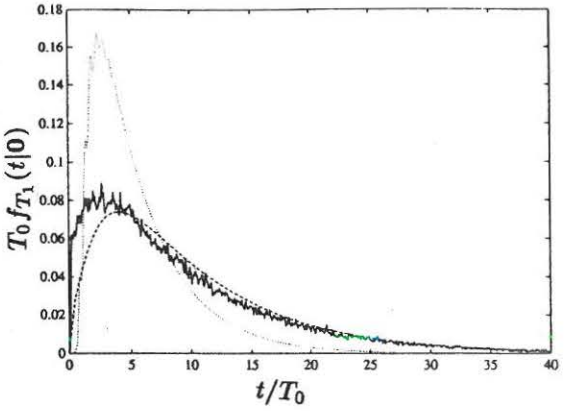

Fig. 8.24: Method 1. First-passage time pdf, $f_{T_1}(t|\mathbf{0})$. Deterministic start, double barrier problem. $b = -a = \sigma_{Z_1,0}$, $\nu = 0.1\omega_0$, $\Delta t = \frac{T_0}{2}$. Uniform mesh: (- - -) simulation, (—) path integration, mesh: (· · ·) white noise excitation.

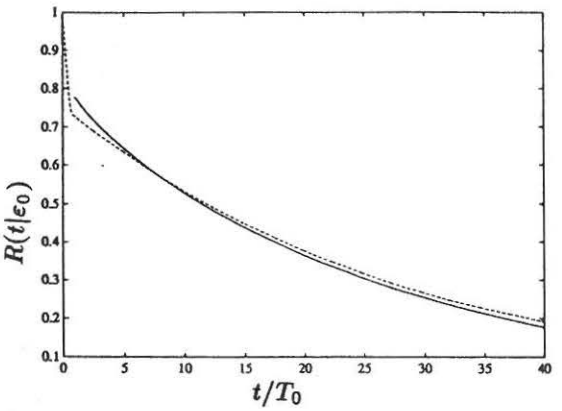

Fig. 8.25: Method 1. Reliability function, $R(t|\mathcal{E}_0)$. Stationary start, single barrier problem. $b = \sigma_{Z_1,0}$, $\nu = 0.01\omega_0$, $\Delta t = T_0$. Uniform mesh: (- - -) simulation, (—) path integration.

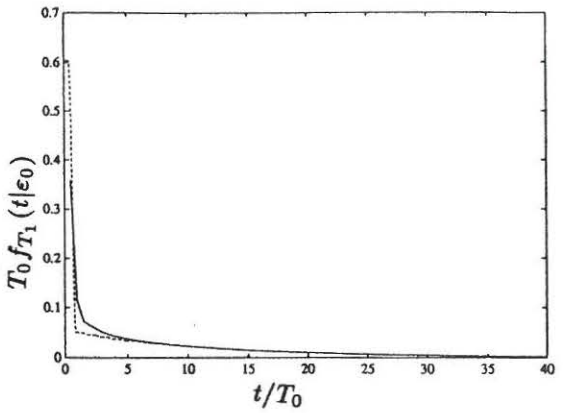

Fig. 8.26: Method 1. First-passage time pdf, $f_{T_1}(t|\mathcal{E}_0)$. Stationary start, single barrier problem. $b = \sigma_{Z_1,0}$, $\nu = 0.01\omega_0$, $\Delta t = T_0$. Uniform mesh: (- - -) simulation, (—) path integration.

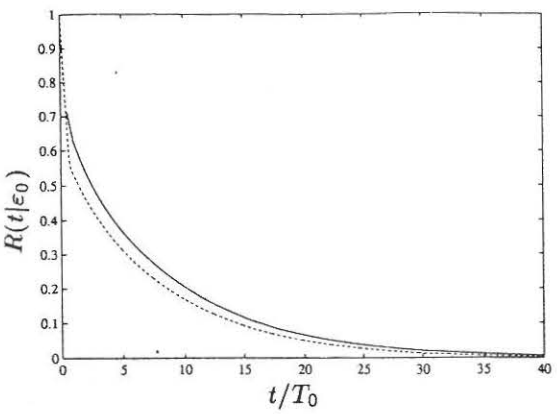

Fig. 8.27: Method 1. Reliability function, $R(t|\mathcal{E}_0)$. Stationary start, single barrier problem. $b = \sigma_{Z_1,0}$, $\nu = 0.1\omega_0$, $\Delta t = \frac{T_0}{2}$. Uniform mesh: (- - -) simulation, (—) path integration.

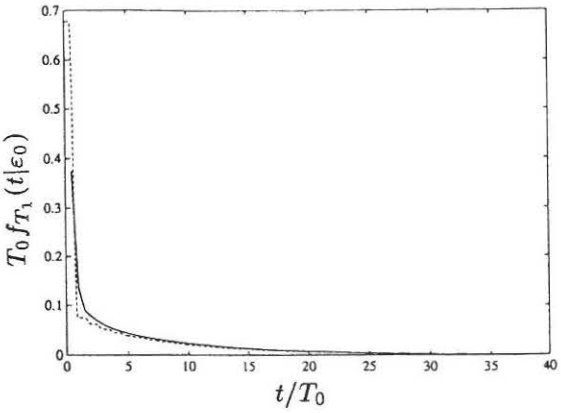

Fig. 8.28: Method 1. First-passage time pdf, $f_{T_1}(t|\mathcal{E}_0)$. Stationary start, single barrier problem. $b = \sigma_{Z_1,0}$, $\nu = 0.1\omega_0$, $\Delta t = \frac{T_0}{2}$. Uniform mesh: (- - -) simulation, (—) path integration.

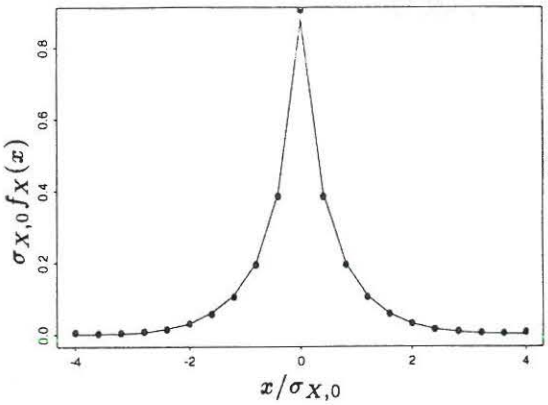

Fig. 8.29: Method 2. Pdf of displacement, $f_X(x)$. Linear scale. $\nu = \frac{0.1}{\pi}\omega_0$, $\Delta t = T_0$. Uniform mesh: (—) simulation, • path integration, iteration, (o) path integration, eigenvector solution.

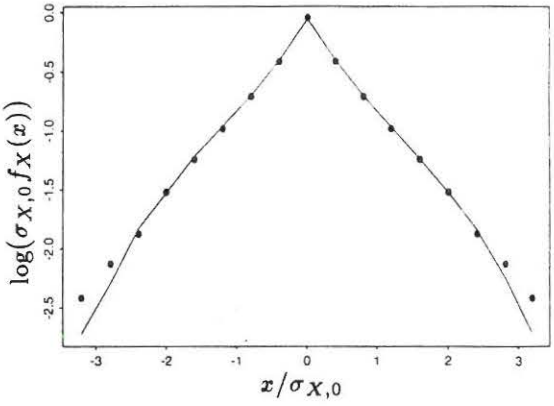

Fig. 8.30: Method 2. Pdf of displacement, $f_X(x)$. Semi-log scale. $\nu = \frac{0.1}{\pi}\omega_0$, $\Delta t = T_0$. Uniform mesh: (—) simulation, • path integration, iteration, (o) path integration, eigenvector solution.

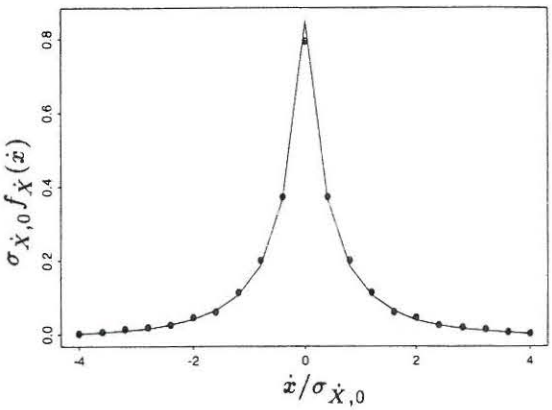

Fig. 8.31: Method 2. Pdf of velocity, $f_{\dot{X}}(\dot{x})$. Linear scale. $\nu = \frac{0.1}{\pi}\omega_0$, $\Delta t = T_0$. Uniform mesh: (—) simulation, • path integration, iteration, (o) path integration, eigenvector solution.

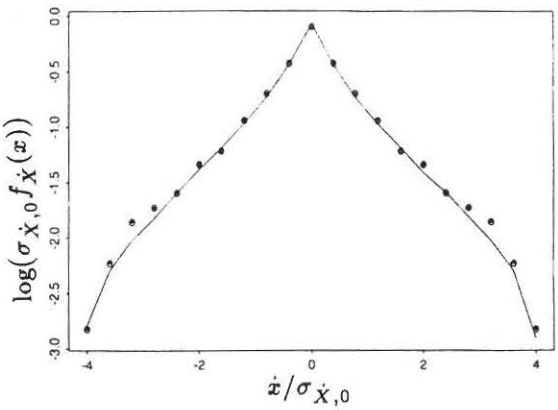

Fig. 8.32: Method 2. Pdf of velocity, $f_{\dot{X}}(\dot{x})$. Semi-log scale. $\nu = \frac{0.1}{\pi}\omega_0$, $\Delta t = T_0$. Uniform mesh: (—) simulation, • path integration, iteration, (o) path integration, eigenvector solution.

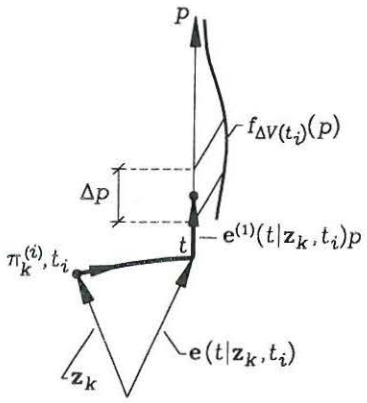

Fig. 8.33: Path integration of compound α-stable Lévy motion driven system. Convection and lumping of probability mass.

The differential equations for a dynamic system driven by α-stable Lévy motion is given by (1.105) with $\{V(t), t \in [t_0, \infty[\}$ signifying an α-stable Lévy motion. During the interval $[\tau, \tau + d\tau[, \tau \in]t_i, t[$, an increment (impulse) of magnitude $dV(\tau)$ is assumed to occur. With the same approximation as assumed in Method 2 for compound Poisson process driven systems, the state vector at the time t from the impulse $dV(\tau)$ can then be written, cf. (8.33), $\mathbf{Z}(t) \simeq \mathbf{e}(t|\mathbf{z}_k, t_i) + \mathbf{e}^{(1)}(t|\mathbf{z}_k, t_i)dV(\tau)$. Since, $\mathbf{Z}(t)$ is independent of τ and depends linearly on $dV(\tau)$, the state vector from all such differential impulses becomes

$$\mathbf{Z}(t) \simeq \mathbf{e}(t|\mathbf{z}_k, t_i) + \mathbf{e}^{(1)}(t|\mathbf{z}_k, t_i)\Delta V(t_i) \quad , \quad \Delta V(t_i) = \int_{t_i}^{t} dV(\tau) = V(t) - V(t_i) \tag{8.36}$$

$\Delta V(t_i)$ in (8.36) is an α-stable random variable $\Delta V(t_i) \sim S_\alpha\big((a\Delta t)^{1/\alpha}, \beta, 0\big)$, cf. section 1.1.3. According to (8.36) the probability mass $\pi_k^{(0)}$ at the node $\mathbf{z}_k$ at the time t_i is then convected to the position $\mathbf{e}(t|\mathbf{z}_k, t_i)$, where it is diffused along the direction of $\mathbf{e}^{(1)}(t|\mathbf{z}_k, t_i)$, according to the probability density function $f_{\Delta V(t_i)}(p)$. At the position $\mathbf{e}(t|\mathbf{z}_k, t_i) + \mathbf{e}^{(1)}(t|\mathbf{z}_k, t_i)p$ a probability mass of magnitude $\pi_k^{(i)} f_{\Delta V(t_i)}(p)\Delta p$ is then lumped as sketched in fig. 8.33.

As shown by (5.19) - (5.23) a system driven by an Erlang renewal process can be reduced to an equivalent Poisson driven system at the expense of the introduction of a number of auxiliary state variables in addition to the structural state variables $\mathbf{X}(t)$, which control that only every kth Poisson generated impulse is applied to the structure. This suggests that the devised path integration schemes for compound Poisson driven systems can be applied also to compound Erlang driven systems as well. However, a modification of the convection and diffusion scheme is needed to ensure that only every kth Poisson impulse causes a convection and diffusion in the mesh. A modification of the Method

2 path integration scheme has been devised by Iwankiewicz and Nielsen [8.15], and will be explained in detail in what follows.

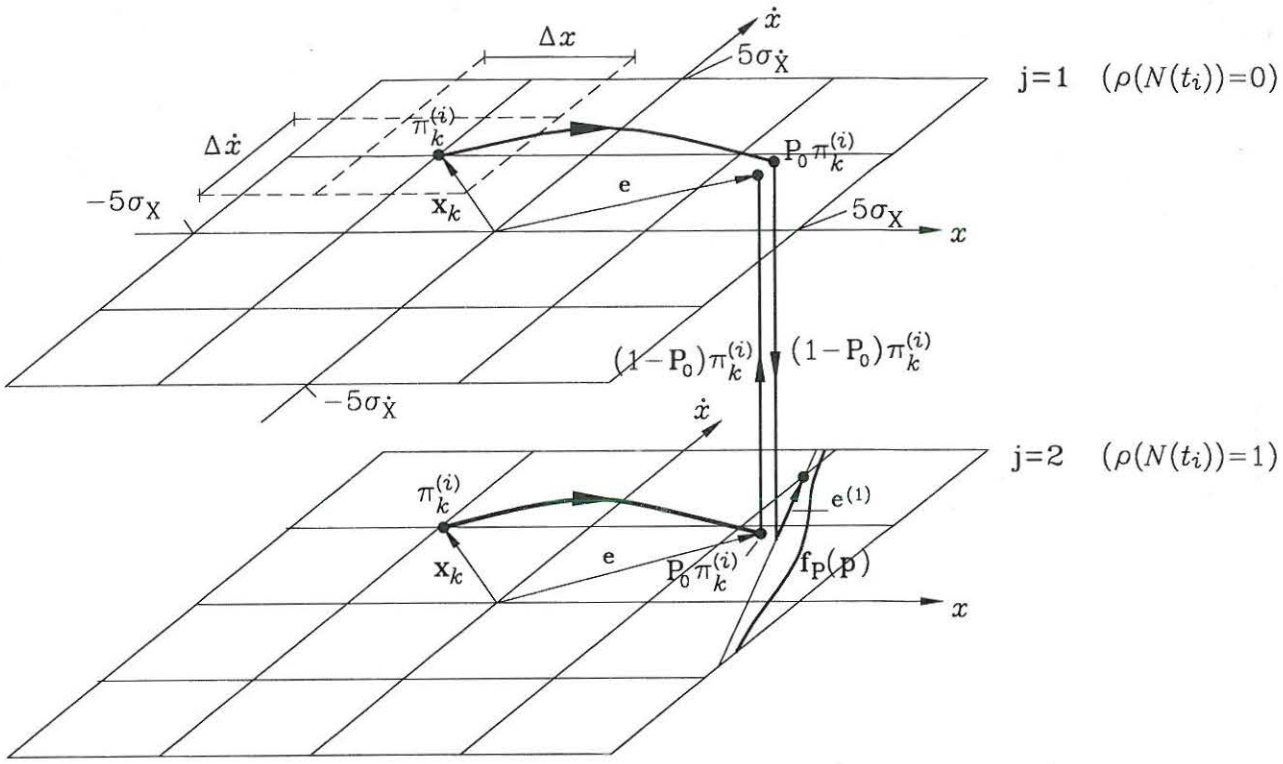

Fig. 8.34. Discretization of the state space for the case $k = 2$. Convection and lumping of the probability mass.

The function $\rho(N(t))$ defined by (5.7), (5.8) repeats itself periodically, attaining the value $\rho(N(t)) = 0$ during the first $k-1$ Poisson events, followed by a value $\rho(N(t)) = 1$ in the kth event. To each Poisson event in the cycle a path integration mesh is defined for convection and diffusion calculation in the discretized structural state space variables, so totally k such meshes should be pointed out, as illustrated in fig. 8.34 for the case $k = 2$ for a 2-dimensional structural sample space.

Assume that the system at the time t_i is in the structural state $\mathbf{X}(t_i) = \mathbf{x}_k$ after the $(j-1)$th Poisson event in the sequence has occurred, $j = 1, \ldots, k-1$. During the interval $]t_i, t]$ the probability mass $\pi_k^{(i)}$ of being at the node $\mathbf{x}_k$ of mesh j at the time t_i is convected to the position $\mathbf{e}(t|\mathbf{x}_k, t_i)$ according to the 0th order term in (8.33). At this position a probability mass of magnitude $\pi_k^{(i)} P_0$ is lumped, where $P_0 = P_0(t, t_i)$ signifies the probability that no Poisson impulse has occurred in $]t_i, t]$. If $j < k-1$ the remaining probability mass $\pi_k^{(i)}(1-P_0)$ is transferred to the succeeding mesh $j+1$, where it is lumped at the same position $\mathbf{e}(t|\mathbf{x}_k, t_i)$. However, if $j = k-1$ the probability mass $\pi_k^{(i)}(1-P_0)$ is distributed along the line $\mathbf{e}^{(1)}(t|\mathbf{x}_k, t_i)$ in the mesh $j = k$, according to the probability density function $f_P(p)$ as in the original formulation of Method 2. In either case the probability mass $\pi_k^{(i)}(1-P_0)$ is attached to the succeeding mesh $j+1$, because

the distribution is performed on the condition that one extra Poisson event has occured during $]t_i, t]$. If the system at the time t_i is in the structural state $\mathbf{X}(t_i) = \mathbf{x}_k$ after the $(k-1)$th Poisson event in the sequence has occurred, the probability mass $\pi_k^{(i)} P_0$ is still lumped at the position $\mathbf{e}(t|\mathbf{x}_k, t_i)$ of mesh $j = k$. The remaining probability mass is transferred to mesh $j = 1$, where it is lumped at the position $\mathbf{e}(t|\mathbf{x}_k, t_i)$ to start a new sequence. As previously, all lumped probability masses are finally redistributed to the adjacent grid points in all k meshes.

As specified by (6.21) and (6.22) the total number of auxiliary state variables amounts to $k-1$, so the dimension of the state vector $\mathbf{Z}(t)$ of the integrated dynamic system becomes $n+k-1$. Appearently, using N cells per state variable in the discretized mesh, the number of states of the Markov chain becomes $(N+1)^{n+k-1}$. However, the number of states of the described path integration scheme only amounts to $k(N+1)^n$, i.e. the growth is linear with k rather than exponential.

Example 8.3: Duffing oscillator subjected to compound Erlang process with k=2

The Duffing oscillator of example 8.1 is considered, when exposed to a stationary compound Erlang process with $k = 2$. The following system data are used $\zeta = 0.01$, κ. The strength of the impulses is assumed to be zero-mean normally distributed, $P \sim N(0, \sigma_P^2)$ with the variance chosen, so $\frac{\nu}{k}\frac{\sigma_P^2}{4\zeta} = 1$, corresponding to the stationary standard deviations $\sigma_{X,0} = \sigma_{\dot{X},0} = 1$ of a linear oscillator exposed to an equivalent Gaussian white noise. Only the case $k = 2$ is considered with the following three values of $\frac{\nu}{k\omega_0}$

$$\frac{\nu}{k\omega_0} = \begin{cases} 0.01 \\ 0.10 \\ 1.00 \end{cases} \quad , \quad \frac{\Delta t}{T_0} = \begin{cases} 1.00 \\ 0.20 \\ 0.05 \end{cases} \quad \Rightarrow \quad \nu\Delta t = \begin{cases} 0.126 \\ 0.251 \\ 0.628 \end{cases} \tag{8.37}$$

The indicated values of $\frac{\nu}{k\omega_0}$ make comparison possible to the compound Poisson cases considered in example 8.2. Again the indicated arrival rates may be categorized as the cases of sparse, medium level and dense pulse arrival rates. The corresponding transition time intervals, Δt, have been selected to meet the upper bound requirement (8.26). Obviously, the lower bound $\frac{\Delta t}{T_0} \geq \frac{1}{2\pi}$, as stated subsequent to (8.27), is not met by the specification of Δt for the case of sparse pulses. Hence, rather poor results are to be expected in this case. The path integration has been performed by a uniform 44×44 mesh with the limits $[-5\sigma_{Z_1,0}, 5\sigma_{X,0}] \times [-5\sigma_{\dot{X},0}, 5\sigma_{\dot{X},0}]$. The stationary marginal pdfs of $X(\infty)$ and $\dot{X}(\infty)$ were estimated from (8.5) with start in the origin of the mesh $j = 1$ using 60 iterations until stationarity. Again, comparison has been made with the results obtained by Monte Carlo simulation. The stationary marginal pdfs were obtained by ergodic sampling, using a time-series of the length $4000000T_0$, which was obtained by numerical integration of the equations of motions by means of a 4th order Runge-Kutta scheme with the time-step $\frac{T_0}{40}$. The generation of the underlying compound Erlang process, and the succeeding numerical integration procedure, was performed as explained in example 5.2. The sampling was performed with the same class-width as applied in the path integration scheme at the end of each integration time-step. Sampling was not initiated until the elapse of an initial transient phase of length $200T_0$.

$\frac{\nu}{k\omega_0}$	$\frac{\Delta t}{T_0}$	$\sigma_X(\infty)$		$\sigma_{\dot{X}}(\infty)$	
		sim.	num.	sim.	num.
0.01	1.00	0.69938	0.70703	0.99768	1.00295
0.10	0.20	0.75587	0.79073	1.00138	1.06577
1.00	0.05	0.76174	0.64546	1.00151	0.80115

Table 8.1: Stationary standard deviations $\sigma_X(\infty)$, $\sigma_{\dot{X}}(\infty)$ of displacement and velocity response as a function of $\frac{\nu}{k\omega_0}$.

In table 3.5 the predictions of the stationary standard deviations obtained from path integration in comparison to those of Monte Carlo simulation are shown. As seen, the results are excellent in the case of sparse pulses $\frac{\nu}{k\omega_0} = 0.01$, they are still quite good in the case of medium level pulse arrivals, $\frac{\nu}{k\omega_0} = 0.10$, but they are not satisfactory for $\frac{\nu}{k\omega_0} = 1.00$.

Below, in figs. 8.35 - 8.38 the stationary marginal probability density functions of the displacement and the velocity for the case of sparse pulse arrivals, $\frac{\nu}{k\omega_0} = 0.01$ are shown, with the non-dimensional transition time interval $\frac{\Delta t}{T_0} = 1.0$. To emphasize on the tails, the results have been indicated both in linear and semi-logarithmic scale. As seen, the agreement with Monte Carlo simulation is very good. Similarly to the comparable Poisson driven system in example 8.2 the marginal pdfs reveal pronounced peaks at the origin in case of sparse pulse arrivals.

Figs. 8.39 and 8.40 show the prediction of path integration method in semi-logarithmic scale for the case $\frac{\nu}{k\omega_0} = 0.01$ for various values of the non-dimensional transition time interval $\frac{\Delta t}{T_0}$. The results for $\frac{\Delta t}{T_0} = 5.0$ are not very good, because the upper bound criterion (8.26) has been violated in this case. The case $\frac{\Delta t}{T_0} = 0.2$, which is close to the lower bound for allowable transition time intervals, still provides good results. However, the best results are obtained for $\frac{\Delta t}{T_0} = 1.0$, which is well within the admissible interval for the transition time interval.

Figs. 8.41 - 8.44 show the corresponding results for the stationary marginal probability density functions of the displacement and the velocity for the case of medium level pulse arrivals $\frac{\nu}{k\omega_0} = 0.1$ with the non-dimensional transition time interval $\frac{\Delta t}{T_0} = 0.2$, which is close to the acceptable lower limit for the transition time interval. As expected the results are not so good as those of the previous case of sparse impulses.

Figs. 8.45 and 8.46 show the dependence of the path integration results on $\frac{\Delta t}{T_0}$ for the same case. The results for $\frac{\Delta t}{T_0} = 1.0$ are unacceptably poor, because the upper bound criterion has been violated in this case. The cases $\frac{\Delta t}{T_0} = 0.2$ and $\frac{\Delta t}{T_0} = 0.1$ both provide more accurate results. However, in the latter case a peak is predicted at the origin, which is not present in the simulation result. This may be attributed to the application of too small transition intervals.

Finally, figs. 8.47 - 8.50 show the results for the stationary marginal probability density functions of the displacement and the velocity for the case of dense pulse arrivals, $\frac{\nu}{k\omega_0} = 1.0$, with the non-dimensional transition time interval $\frac{\Delta t}{T_0} = 0.05$. Due to the application of too small transition time intervals the path integration results are neither qualitatively nor quantitatively in agreement with simulation results. Again a peak is predicted at the origin, which is not present in the simulation results.

The dependence on $\frac{\Delta t}{T_0}$ for the same case is shown in figs. 8.51 and 8.52. Although none of the cases provide acceptable results, the accuracy is much better for the cases $\frac{\Delta t}{T_0}$=0.05 and 0.1, than for the case $\frac{\Delta t}{T_0} = 0.02$.

The example demonstrates the applicability of path integration technique for a Duffing oscillator subjected to impulses driven by an Erlang renewal process of order $k = 2$. The applied path integration scheme was a modification of the Method 2 for compound Poisson driven systems. Again, it has been demonstrated that the path integration method provides accurate results for the case of sparse pulses, whereas unaccaptable results are obtained if the non-dimensional transition interval $\frac{\Delta t}{T_0}$ is either too small or too large.

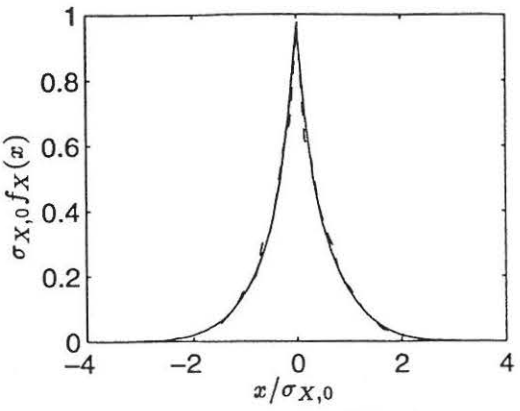

Fig. 8.35: Stationary pdf of displacement, $f_X(x)$. Linear scale. $k = 2$, $\frac{\nu}{k\omega_0} = 0.01$, $\frac{\Delta t}{T_0} = 1.0$. Uniform mesh: (—) simulation, (- - -) path integration.

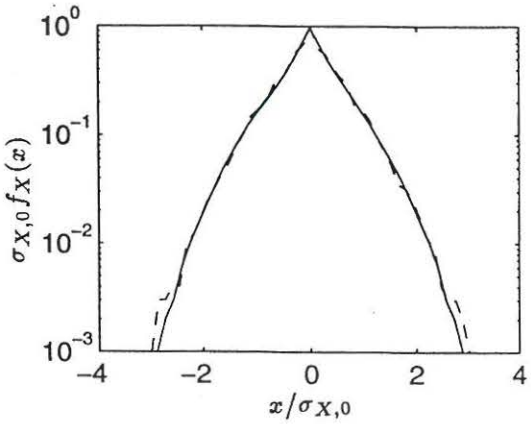

Fig. 8.36: Stationary pdf of displacement, $f_X(x)$. Semi-log scale. $k = 2$, $\frac{\nu}{k} = 0.01\omega_0$, $\frac{\Delta t}{T_0} = 1.0$. Uniform mesh: (—) simulation, (- - -) path integration.

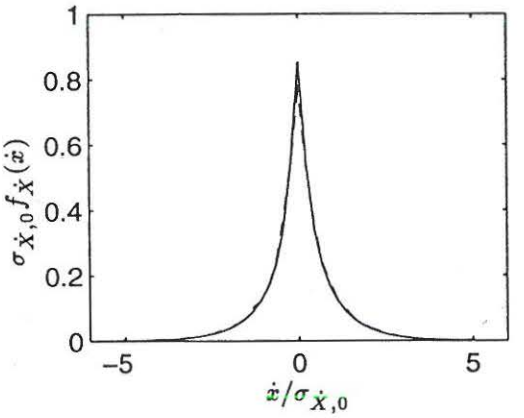

Fig. 8.37: Stationary pdf of velocity, $f_{\dot{X}}(\dot{x})$. Linear scale. $k = 2$, $\frac{\nu}{k\omega_0} = 0.01$, $\frac{\Delta t}{T_0} = 1.0$. Uniform mesh: (—) simulation, (- - -) path integration.

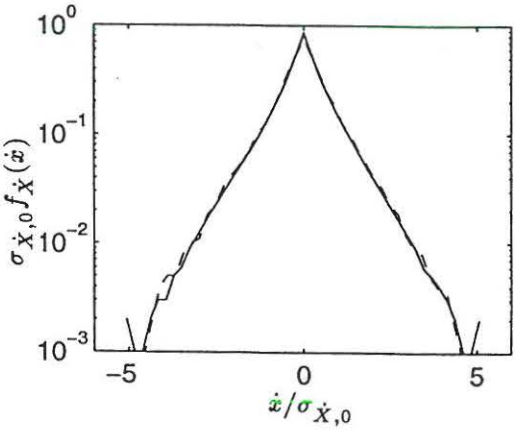

Fig. 8.38: Stationary pdf of velocity, $f_{\dot{X}}(\dot{x})$. Semi-log scale. $k = 2$, $\frac{\nu}{k} = 0.01\omega_0$, $\frac{\Delta t}{T_0} = 1.0$. Uniform mesh: (—) simulation, (- - -) path integration.

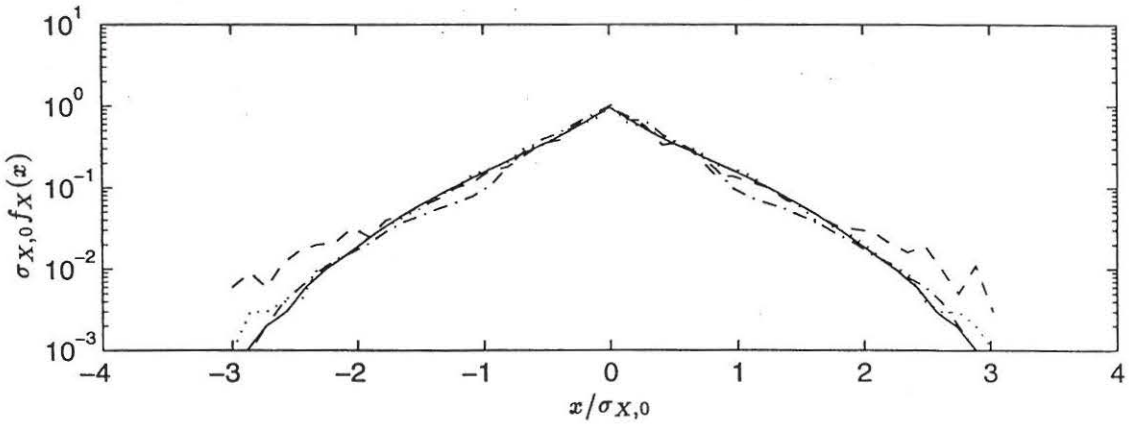

Fig. 8.39: Dependence of path integration results on the length of transition time interval Δt. Stationary pdf of displacement, $f_X(x)$. Semi-log scale. $k = 2$, $\frac{\nu}{k\omega_0} = 0.01$, uniform mesh. (—): simulation, (- - -): $\frac{\Delta t}{T_0} = 5.0$, $(\cdots)$: $\frac{\Delta t}{T_0} = 1.0$, (-·-·-): $\frac{\Delta t}{T_0} = 0.2$.

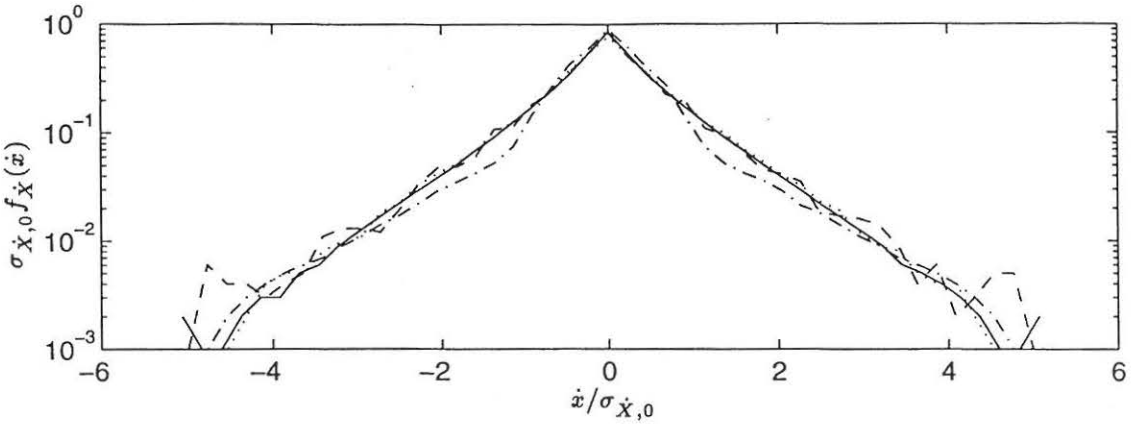

Fig. 8.40: Dependence of path integration results on the length of transition time interval Δt. Stationary pdf of velocity, $f_{\dot{X}}(\dot{x})$. Semi-log scale. $k = 2$, $\frac{\nu}{k\omega_0} = 0.01$, uniform mesh. (—): simulation, (- - -): $\frac{\Delta t}{T_0} = 5.0$, $(\cdots)$: $\frac{\Delta t}{T_0} = 1.0$, (-·-·-): $\frac{\Delta t}{T_0} = 0.2$.

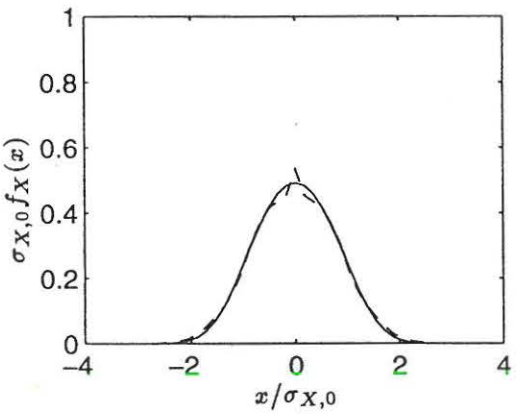

Fig. 8.41: Stationary pdf of displacement, $f_X(x)$. Linear scale. $k = 2$, $\frac{\nu}{k\omega_0} = 0.1$, $\frac{\Delta t}{T_0} = 0.2$. Uniform mesh: (—) simulation, (- - -) path integration.

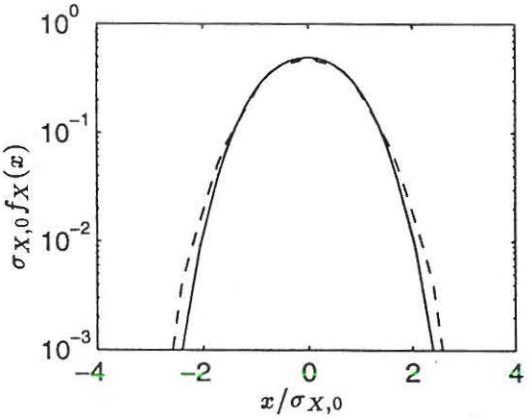

Fig. 8.42: Stationary pdf of displacement, $f_X(x)$. Semi-log scale. $k = 2$, $\frac{\nu}{k\omega_0} = 0.1$, $\frac{\Delta t}{T_0} = 0.2$. Uniform mesh: (—) simulation, (- - -) path integration.

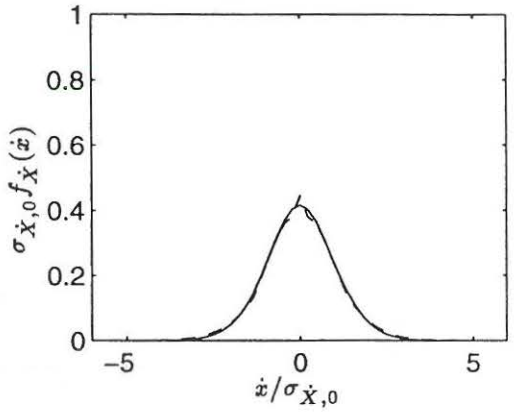

Fig. 8.43: Stationary pdf of velocity, $f_{\dot{X}}(\dot{x})$. Linear scale. $k = 2$, $\frac{\nu}{k\omega_0} = 0.1$, $\frac{\Delta t}{T_0} = 0.2$. Uniform mesh: (—) simulation, (- - -) path integration.

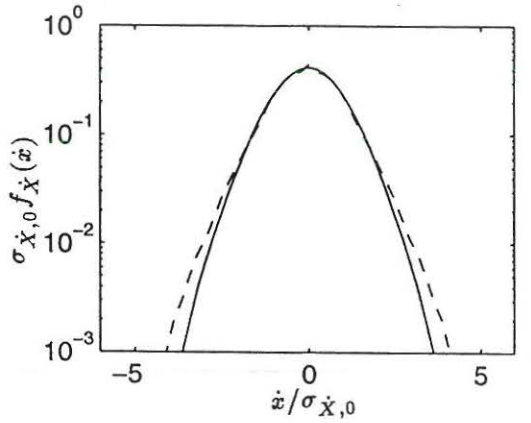

Fig. 8.44: Stationary pdf of velocity, $f_{\dot{X}}(\dot{x})$. Semi-log scale. $k = 2$, $\frac{\nu}{k\omega_0} = 0.1$, $\frac{\Delta t}{T_0} = 0.2$. Uniform mesh: (—) simulation, (- - -) path integration.

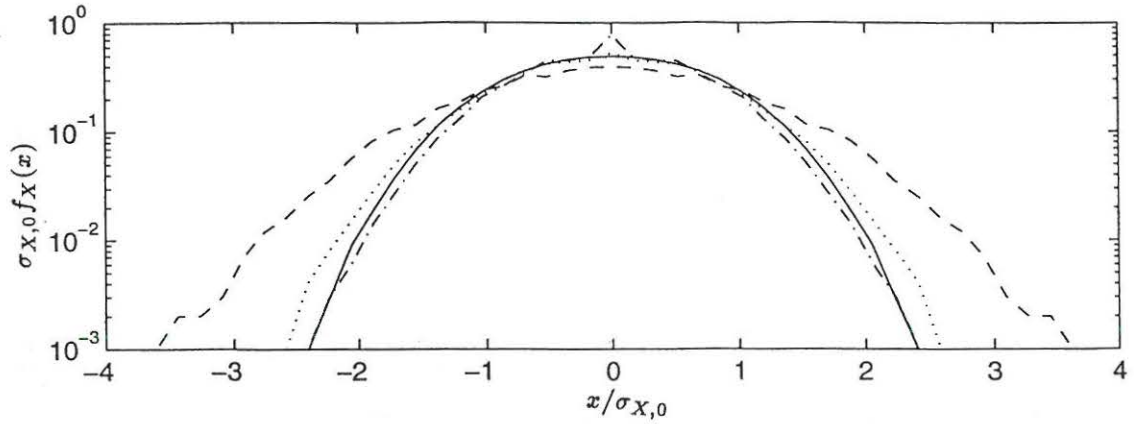

Fig. 8.45: Dependence of path integration results on the length of transition time interval Δt. Stationary pdf of displacement, $f_X(x)$. Semi-log scale. $k = 2$, $\frac{\nu}{k\omega_0} = 0.1$, uniform mesh.
(—): simulation, (- - -): $\frac{\Delta t}{T_0} = 1.0$, $(\cdots)$: $\frac{\Delta t}{T_0} = 0.2$, (-·-·-): $\frac{\Delta t}{T_0} = 0.1$.

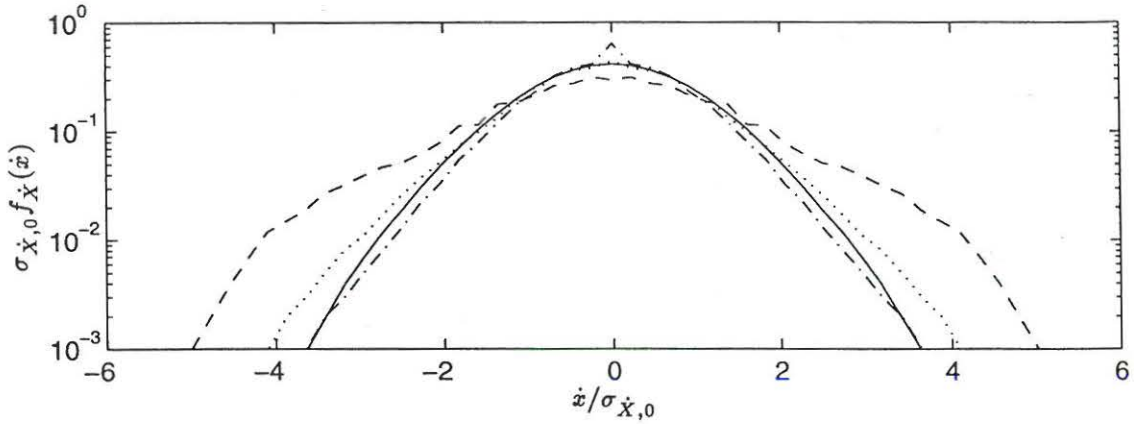

Fig. 8.46: Dependence of path integration results on the length of transition time interval Δt. Stationary pdf of velocity, $f_{\dot{X}}(\dot{x})$. Semi-log scale. $k = 2$, $\frac{\nu}{k\omega_0} = 0.1$, uniform mesh.
(—): simulation, (- - -): $\frac{\Delta t}{T_0} = 1.0$, $(\cdots)$: $\frac{\Delta t}{T_0} = 0.2$, (-·-·-): $\frac{\Delta t}{T_0} = 0.1$.

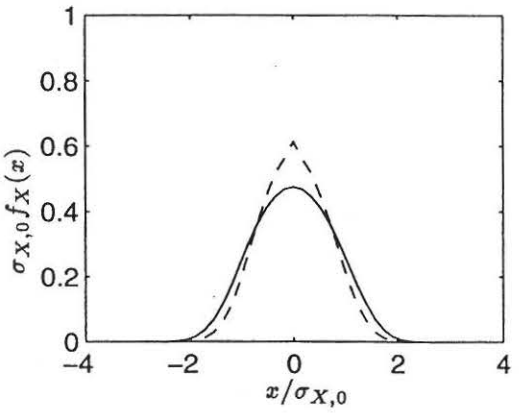

Fig. 8.47: Stationary pdf of displacement, $f_X(x)$. Linear scale. $k = 2$, $\frac{\nu}{k\omega_0} = 1.0$, $\frac{\Delta t}{T_0} = 0.05$. Uniform mesh: (—) simulation, (- - -) path integration.

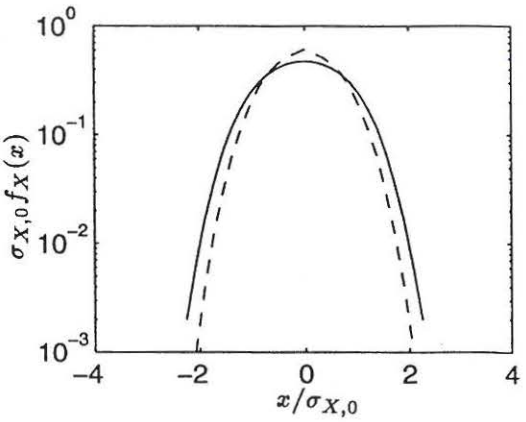

Fig. 8.48: Stationary pdf of displacement, $f_X(x)$. Semi-log scale. $k = 2$, $\frac{\nu}{k\omega_0} = 1.0$, $\frac{\Delta t}{T_0} = 0.05$. Uniform mesh: (—) simulation, (- - -) path integration.

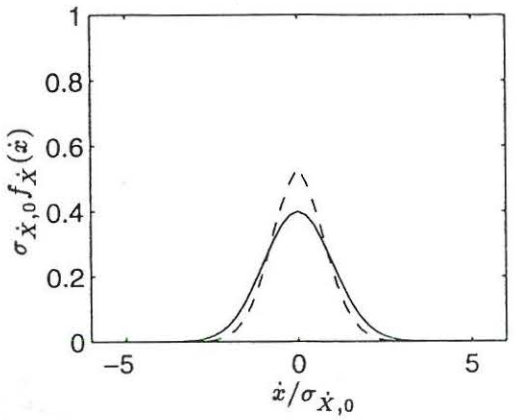

Fig. 8.49: Stationary pdf of velocity, $f_{\dot{X}}(\dot{x})$. Linear scale. $k = 2$, $\frac{\nu}{k\omega_0} = 1.0$, $\frac{\Delta t}{T_0} = 0.05$. Uniform mesh: (—) simulation, (- - -) path integration.

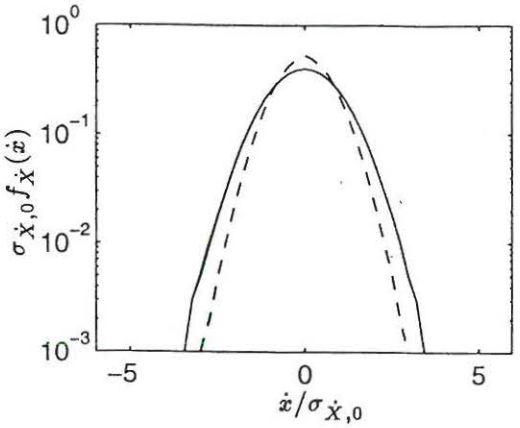

Fig. 8.50: Stationary pdf of velocity, $f_{\dot{X}}(\dot{x})$. Semi-log scale. $k = 2$, $\frac{\nu}{k\omega_0} = 1.0$, $\frac{\Delta t}{T_0} = 0.05$. Uniform mesh: (—) simulation, (- - -) path integration.

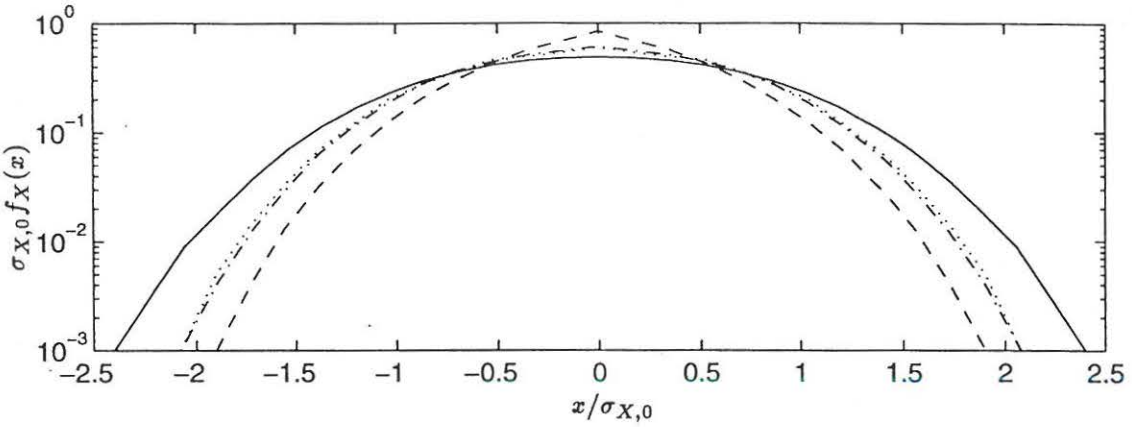

Fig. 8.51: Dependence of path integration results on the length of transition time interval Δt. Stationary pdf of displacement, $f_X(x)$. Semi-log scale. $k = 2$, $\frac{\nu}{k\omega_0} = 1.0$, uniform mesh.
(—): simulation, (- - -): $\frac{\Delta t}{T_0} = 0.02$, (· · ·): $\frac{\Delta t}{T_0} = 0.1$, (-·-·-): $\frac{\Delta t}{T_0} = 0.05$.

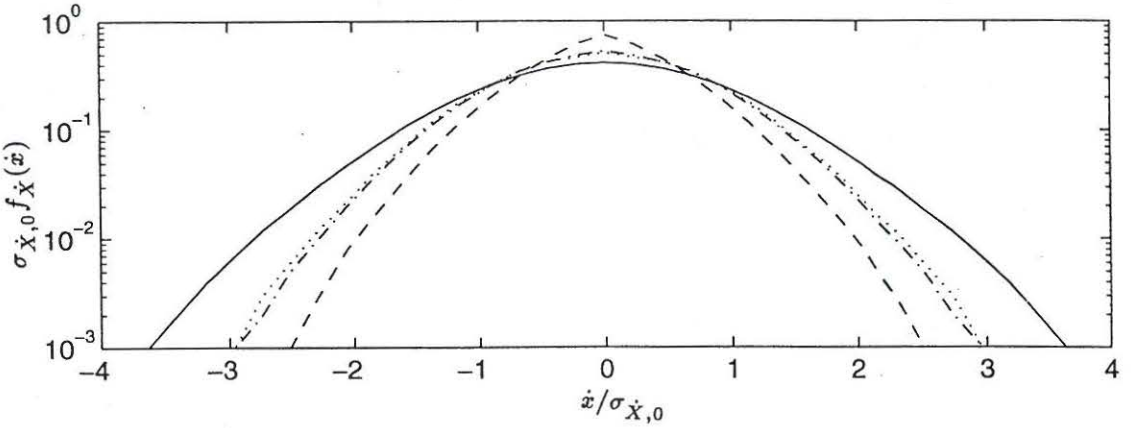

Fig. 8.52: Dependence of path integration results on the length of transition time interval Δt. Stationary pdf of velocity, $f_{\dot{X}}(\dot{x})$. Semi-log scale. $k = 2$, $\frac{\nu}{k\omega_0} = 1.0$, uniform mesh.
(—): simulation, (- - -): $\frac{\Delta t}{T_0} = 0.02$, (· · ·): $\frac{\Delta t}{T_0} = 0.1$, (-·-·-): $\frac{\Delta t}{T_0} = 0.05$.

8.3 Concluding remarks

In this chapter it has been demonstrated how the Chapman-Kolmogorov forward integro-differential equation for the simple physical systems can be solved by means of the cell-to-cell mapping techniques (path integration).

For white noise driven systems the method is based on piecewise linearization and the related local Gaussianity of the conditional response. Four different linearization schemes have been presented. Of these, the running mean linearization scheme, and the equivalent linearization scheme in the expected least square sense are especially attractive, since they make possible the applicability of significantly larger transition time intervals than the other linarization schemes. This turns out to be useful in non-stationary stochastic response calculations or in reliability problems, where the convection and diffusion of the probability mass have to be obtained by mutual iteration of the evolutionary equations for the Markov chain. For the stationary distributions this objection is less important, since these can be found upon solving a linear eigenvalue problem with the known eigenvalue $\lambda = 1$. A numerical example, where the equivalent linearization approach was used to a Duffing oscillator, demonstrates that the accurate results for the marginal probability density functions and for their tails up to a distance of 4 standard deviations of the corresponding linear oscillator can be obtained with the computation mesh as crude as 20×20, using a time step of $T_0/4$. In a deterministic start single barrier first-passage time problem it is demonstrated that reliability problem can be handled as well. However the probability of failure during the first periods of the transition of the probability mass from its initial singularity cannot be captured very well with the applied crude calculation mesh, and adjusted large transition interval of $\Delta t = T_0/2$.

Next the application of the method to compound Poisson process driven systems has been demonstrated. The basis of the method is an expansion of the transition probability density function in terms of the transition probability density functions conditional upon arrivals of exactly n impulses during the transition time interval Δt, the expansion being truncated at the second term, assuming that at most 1 impulse can arrive during Δt. The error of truncation of the series is shown to be of the order $O\left((\nu\Delta t)^2\right)$. Then, for a certain transition interval the method will work the best for small mean arrival rates ν, which is in contrast to all other known numerical solution methods. Two alternative convection and lumping schemes for the probability mass, on condition of exactly 1 impulse arrival during the transition time interval have been indicated. Especially the latter of these turns out to be simple and fast to use. In two succeeding numerical methods to the lumping schemes were applied to a Duffing oscillator. Still using basically a 20×20 mesh, with local refinements in case of very sparse pulses, the capability of the method in predicting the marginal probability densities and their tails. It was demonstrated that the basic assumption of the methods, i.e. $\nu\Delta t << 1$ is, in practice, fulfilled within acceptable accuracy (less than 1.8 %), if only $\nu\Delta t < 0.2$. The necessary calculation time of using the 2nd convection scheme and of using Monte Carlo simulation method, on condition of attaining comparable accuracy in the tails of distributions at a distance of 4 standard deviations (of the corresponding linear oscillator subjected to a Gaussian white noise) turned out to be 31 sec. and 13 950 sec., respectively. The 1st lumping scheme was also applied to a symmetric double barrier

first-passage time problem for both the deterministic and the stochastic start problem. Again the problems of convection and lumping in the initial phases of first-passages with the applied crude 20×20 mesh were demonstrated.

Finally, a convection and lumping scheme for the systems driven by α-stable Lévy motion have been indicated to be of comparable quality as the simplest linearization scheme for Gaussian white noise driven systems. No numerical example has been given in this case, because no simulation results are available for comparison.

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8.5 Example problems

8.1 Consider a quadratic mesh as shown in fig. 9.7. Derive a path integration scheme based on an isoparametric interpolation among the adjacent nodal values with the shape functions $N_i(y, \dot{y})$ as given by (9.58), (9.59), and calculate the equivalents to (8.3) and (8.4).

8.2 Derive the results (8.18)-(8.21) from the basic forms (8.12)-(8.21) for the system defined by (8.17).

CHAPTER 9

PETROV-GALERKIN METHODS TO SOLVE THE FORWARD AND BACKWARD INTEGRO-DIFFERENTIAL CHAPMAN-KOLMOGOROV EQUATIONS

Consider the following initial-boundary value problem for the one-dimensional Fokker-Planck equation, cf. (2.71), (2.72), (6.28), which will be formulated in a slightly more general way in the following

$$\left.\begin{array}{lll} \frac{\partial}{\partial t}f(z,t) = -\frac{\partial}{\partial z}\big(c(z,t)f(z,t)\big) + \frac{1}{2}\frac{\partial^2}{\partial z^2}\big(D(z,t)f(z,t)\big) & , \forall\, t \in]t_0,t_1] & , \forall\, z \in]a,b[\\ f(z,t_0) = f_0(z) & & , \forall\, z \in]a,b[\\ f(a,t) = f(b,t) = 0 & , \forall\, t \in]t_0,t_1] & \end{array}\right\} \tag{9.1}$$

$f_0(z)$ is the initial value at the time t_0 for the unknown function $f(z,t)$. Further, absorbing boundary conditions are prescribed at $z = a$ and $z = b$. Let $\delta v(z)$ be a piecewise continuously differentiable function at least of the 2nd order, which fulfils the boundary conditions $v(a) = v(b) = 0$. Upon multiplying the differential equation in (9.1) with $\delta v(z)$, and performing integration by parts the following weak formulation of (9.1) is obtained

$$\int_a^b \delta v(z)\frac{\partial}{\partial t}f(z,t)dz = \int_a^b f(z,t)\Big[c(z,t)\frac{d}{dz}\delta v(z) + \frac{1}{2}D(z,t)\frac{d^2}{dz^2}\delta v(z)\Big]dx =$$

$$\int_a^b f(z,t)\mathcal{K}_{z,t}^T[\delta v(z)]dz \tag{9.2}$$

where $\mathcal{K}_{z,t}^T[f(z,t)] = c(z,t)\frac{\partial}{\partial z}f(z,t) + \frac{1}{2}D(z,t)\frac{\partial^2}{\partial z^2}f(z,t)$ is the backward Kolmogorov operator (2.73). Hence, the requirements on differentiability and boundary conditions for the variational field $\delta v(z)$ insure that the Fokker-Planck operator $\mathcal{K}_{z,t}[\cdot]$ and the backward Kolmogorov operator $\mathcal{K}_{z,t}^T[\cdot]$ become adjoint operators over the interval $[a,b]$, i.e.

$$\int_a^b \delta v(z)\mathcal{K}_{z,t}[f(z,t)]dz = \int_a^b f(z,t)\mathcal{K}_{z,t}^T[\delta v(z)]dz \tag{9.3}$$

The previous results can now be generalized to the following initial-boundary value problem for the multi-dimensional forward integro-differential Chapman-Kolmogorov

equation (6.28)

$$\left.\begin{array}{lll} \frac{\partial}{\partial t} f(\mathbf{z},t) = \mathcal{K}_{\mathbf{z},t}[f(\mathbf{z},t)] & , \forall\, t \in]t_0,t_1] & , \forall\, \mathbf{z} \in S_t \\ f(\mathbf{z},t_0) = f_0(\mathbf{z}) & & , \forall\, \mathbf{z} \in S_{t_0} \\ f(\mathbf{z},t) = 0 & , \forall\, t \in]t_0,t_1] & , \forall\, \mathbf{z} \in \partial S_t^{(0)} \cup \partial S_t^{(2)} \end{array}\right\} \tag{9.4}$$

S_t is the solution set at the time t, bounded by the surface $\partial S_t = \partial S_t^{(0)} \cup \partial S_t^{(1)} \cup \partial S_t^{(2)}$. $\partial S_t^{(2)}$ is the non-accessible (natural) part of the boundary, whereas the accessible boundary is made up of the exit part $\partial S_t^{(1)}$ and the entrance part $\partial S_t^{(0)}$, see fig. 2.1.

In order to derive the weak counterpart of (9.4), consider a variational field $\delta v(\mathbf{z})$ sufficiently smooth, and fulfilling the necessary boundary conditions making valid the following generalization of (9.3)

$$\int_{S_t} \delta v(\mathbf{z}) \mathcal{K}_{\mathbf{z},t}[f(\mathbf{z},t)] d\mathbf{z} = \int_{S_t} f(\mathbf{z},t) \mathcal{K}^T_{\mathbf{z},t}[\delta v(\mathbf{z})] d\mathbf{z} \tag{9.5}$$

Generally, (9.5) is valid if the variational field fulfils $\delta v(\mathbf{z}) = 0$ for $\mathbf{z} \in \partial S_t^{(1)}$, i.e. for the part of boundary surface where no boundary conditions need to be specified for $f(\mathbf{z},t)$. $\mathcal{K}_{\mathbf{z},t}[\cdot]$ and $\mathcal{K}^T_{\mathbf{z},t}[\cdot]$ denote the forward- and backward integro-differential operators (2.25) and (2.39). If the differential equation in (9.4) is multiplied by the variational field $\delta v(\mathbf{z})$ followed by an integration over S_t, the following weak formulation of the problem can then be obtained by multiple application of the divergence theorem

$$\int_{S_t} \delta v(\mathbf{z}) \frac{\partial}{\partial t} f(\mathbf{z},t) d\mathbf{z} = \int_{S_t} f(\mathbf{z},t) \mathcal{K}^T_{\mathbf{z},t}[\delta v(\mathbf{z})] d\mathbf{z} \tag{9.6}$$

The following approximate series expansions are assumed for the solution of (9.4) and for the variational field

$$f(\mathbf{z},t) \simeq \sum_{j=1}^{N} f_j(t) N_j(\mathbf{z}) \tag{9.7}$$

$$\delta v(\mathbf{z}) \simeq \sum_{i=1}^{N} \delta v_i V_i(\mathbf{z}) \tag{9.8}$$

$N_j(\mathbf{z})$ is assumed to be piecewise continuous in S_t and to fulfil the same boundary conditions $N_j(\mathbf{z}) = 0$ for any $\mathbf{z} \in \partial S_t^{(0)} \cup \partial S_t^{(2)}$ as does $f(\mathbf{z},t)$. No boundary conditions are prescribed on $\partial S_t^{(1)}$. The setting (9.8) restricts variations to a subspace spanned by the weighting functions $V_i(\mathbf{z})$, which all possess the same differentiability properties

and fulfil the same boundary conditions on ∂S_t as required for $\partial v(\mathbf{z})$. Insertion of (9.7) and (9.8) into (9.6) provides

$$\sum_{i=1}^{N} \delta v_i \left[\sum_{j=1}^{N} M_{ij}(t)\dot{f}_j(t) - \sum_{j=1}^{N} K_{ij}(t) f_j(t) \right] = 0 \tag{9.9}$$

$$M_{ij}(t) = \int_{S_t} V_i(\mathbf{z}) N_j(\mathbf{z}) d\mathbf{z} \tag{9.10}$$

$$K_{ij}(t) = \int_{S_t} N_j(\mathbf{z}) \mathcal{K}^T_{\mathbf{z},t}[V_i(\mathbf{z})] d\mathbf{z} \tag{9.11}$$

where it has been used that $\mathcal{K}^T_{\mathbf{z},t}[\cdot]$ is a linear operator. Notice, that the smoothness assumptions on $N_j(\mathbf{z})$ and $V_i(\mathbf{z})$ imply that the integral on the right-hand side of (9.11) becomes meaningful in ordinary Riemann sense. Since (9.9) must be fulfilled for any variation δv_i the following Euler conditions are derived for the determination of the unknown functions $f_j(t)$

$$\sum_{j=1}^{N} M_{ij}(t)\dot{f}_j(t) - \sum_{j=1}^{N} K_{ij}(t) f_j(t) = 0 \tag{9.12}$$

The ordinary differential equations (9.12) must be solved with suitable initial conditions $f_j(t)$ at the time $t = t_0$. These are derived upon multiplying the initial conditions of (9.4) by $\delta v(\mathbf{z})$ and performing the integration over S_{t_0}. After insertion of (9.7) and (9.8), the variational conditions lead to the following linear equations for the determination of $f_j(t_0)$

$$\sum_{j=1}^{N} M_{ij}(t_0) f_j(t_0) = \int_{S_{t_0}} V_i(\mathbf{z}) f_0(\mathbf{z}) d\mathbf{z} \tag{9.13}$$

Especially, for the problem defined in (9.1), eqs. (9.7), (9.8), (9.10), (9.11) take the form

$$f(z,t) \simeq \sum_{j=1}^{N} f_j(t) N_j(z) \tag{9.14}$$

$$\delta v(z) \simeq \sum_{i=1}^{N} \delta v_i V_i(z) \tag{9.15}$$

$$M_{ij} = \int_{a}^{b} V_i(z) N_j(z) dz \tag{9.16}$$

$$K_{ij}(t) = \int_a^b N_j(z)\Big[c(z,t)\frac{d}{dz}V_i(z) + \frac{1}{2}D(z,t)\frac{d^2}{dz^2}V_i(z)\Big]dz \tag{9.17}$$

Next, consider the boundary value problem (6.32) for the backward integro-differential Chapman-Kolmogorov equation, which will be formulated in the following way

$$\left.\begin{array}{lll} \frac{\partial}{\partial t}f(\mathbf{z},t) + \mathcal{K}^T_{\mathbf{z},t}[f(\mathbf{z},t)] = 0 \quad , & \forall\, t \in [t_0,t_1[\quad , & \forall\, \mathbf{z} \in S_t \\ f(\mathbf{z},t_1) = f_1(\mathbf{z}) & , & \forall\, \mathbf{z} \in S_{t_1} \\ f(\mathbf{z},t) = 0 & , \quad \forall\, t \in [t_0,t_1[\quad , & \forall\, \mathbf{z} \in \partial S_t^{(1)} \cup \partial S_t^{(2)} \end{array}\right\} \tag{9.18}$$

$f_1(\mathbf{z})$ is the terminal value of the unknown function $f(\mathbf{z},t)$, and (9.18) is solved backwards with time.

At the formulation of the weak counterpart of (9.18) the variational field $\delta v(\mathbf{z})$ is chosen to be sufficiently smooth and to fulfil the necessary boundary conditions making the following statement valid

$$\int_{S_t} \delta v(\mathbf{z})\mathcal{K}^T_{\mathbf{z},t}[f(\mathbf{z},t)]d\mathbf{z} = \int_{S_t} f(\mathbf{z},t)\mathcal{K}_{\mathbf{z},t}[\delta v(\mathbf{z})]d\mathbf{z} \tag{9.19}$$

(9.19) turns out to be valid, if the variational field fulfils $\delta v(\mathbf{z}) = 0$ for $\mathbf{z} \in \partial S_t^{(0)}$. Again, the variational field is required to cancel on the part of the surface, where no boundary conditions are prescribed for the function searched for. Next, the differential equation in (9.18) is multiplied by the variational field $\delta v(\mathbf{z})$ followed by an integration over S_t. Use of (9.19) provides the following weak formulation of the problem

$$\int_{S_t} \delta v(\mathbf{z})\frac{\partial}{\partial t}f(\mathbf{z},t)d\mathbf{z} + \int_{S_t} f(\mathbf{z},t)\mathcal{K}^T_{\mathbf{z},t}[\delta v(\mathbf{z})]d\mathbf{z} = 0 \tag{9.20}$$

The expansions (9.7) and (9.8) are still used. However, $N_j(\mathbf{z})$ is now assumed to be piecewise continuous and to fulfil the boundary conditions $N_j(\mathbf{z}) = 0$ for any $\mathbf{z} \in \partial S_t^{(0)} \cup \partial S_t^{(1)}$, and no boundary conditions are prescribed on $\partial S_t^{(2)}$. $V_i(\mathbf{z})$ is assumed to possess the same differentiability properties and to fulfil the same boundary conditions on ∂S_t as required for $\partial v(\mathbf{z})$. Insertion of (9.7) and (9.8) and performance of the variational procedure again lead to (9.12). The "mass" tensor $M_{ij}(t)$ is still given by (9.10), whereas the "stiffness" tensor $K_{ij}(t)$ is given by

$$K_{ij}(t) = -\int_{S_t} N_j(\mathbf{z})\mathcal{K}_{\mathbf{z},t}[V_i(\mathbf{z})]d\mathbf{z} \tag{9.21}$$

(9.12) should now be solved backwards with time with the terminal conditions $f_j(t_1)$ determined from

$$\sum_{j=1}^{N} M_{ij}(t_1) f_j(t_1) = \int_{S_{t_1}} V_i(\mathbf{z}) f_1(\mathbf{z}) d\mathbf{z} \tag{9.22}$$

If $V_i(\mathbf{z}) = N_i(\mathbf{z})$, $i = 1, \ldots, N$, the indicated approaches for solving the forward or backward integro-differential Chapman-Kolmogorov equations (9.4) and (9.18) reduce to the *standard Galerkin variational method.* However, attempts to solve (9.4) or (9.18) in this way may lead to numerical stability problems, if the magnitude of the drift vector $|\mathbf{c}(\mathbf{z}, t)|$ grows without limits in parts of the solution space, as is the case for a white noise excited SDOF oscillator with non-accessible (natural) boundaries where the first component $c_1(y, \dot{y}, t) = \dot{y}$ can be of infinite magnitude, cf. (1.91).

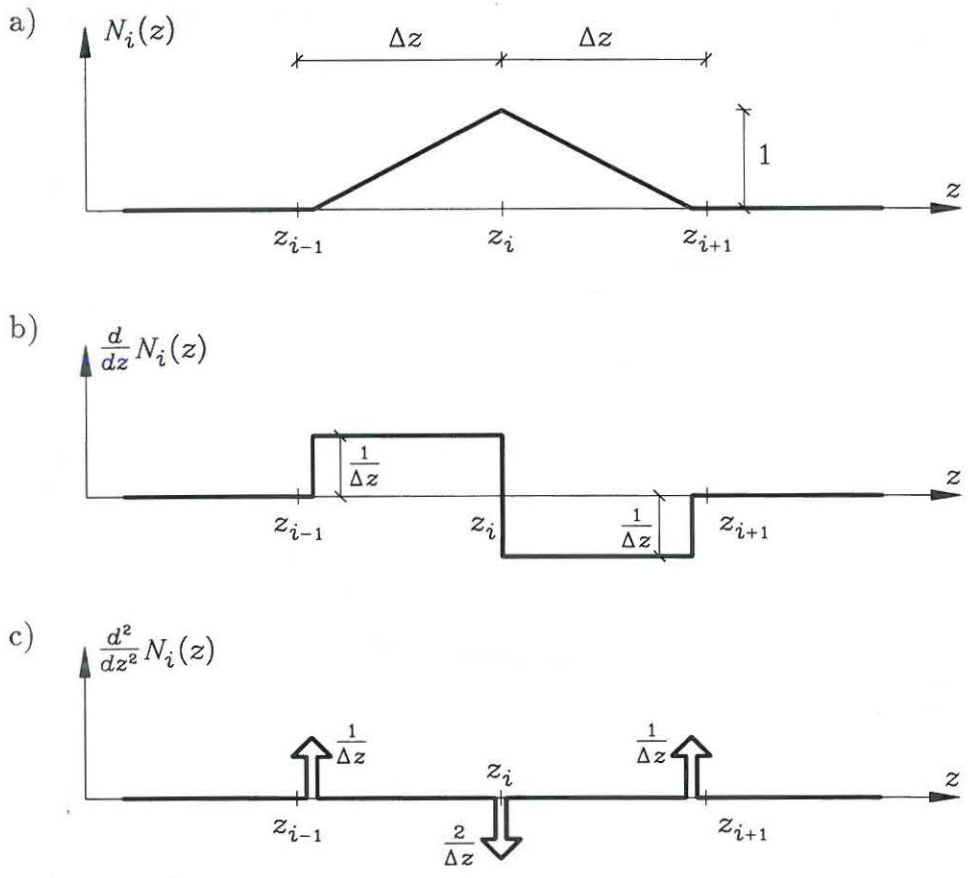

Fig. 9.1: Shape functions for one-dimensional Fokker-Planck equation.
a) $N_i(z)$. b) $\frac{d}{dz} N_i(z)$. c) $\frac{d^2}{dz^2} N_i(z)$.

In order to analyse this problem deeper the system (9.1) is considered again. For ease the drift function c and the diffusion function d are both assumed to be constants as functions of z and t. At the numerical solution the interval $[a, b]$ is divided into

$N+1$ sub-intervals of equal width $\Delta z = \frac{b-a}{N+1}$ with the ith interval given by $[z_{i-1}, z_i]$, $z_i = a + i\Delta z$, $i = 0, 1, \ldots, N+1$. The shape functions $N_i(z)$ will be taken as piecewise linear continuous functions with $f_j(t)$ representing the nodal values of $f(z,t)$. $N_i(z)$ then appears as shown in fig. 9.1a. Obviously, the boundary conditions $N_i(a) = N_i(b) = 0$ are fulfilled for $i = 1, \ldots, N$. In case of the standard Galerkin method, $V_i(z) = N_i(z)$, (9.16) and (9.17) then become

$$\mathbf{M} = \left[\int_{z_{i-1}}^{z_{i+1}} N_i(z) N_j(z) dz \right] = \frac{\Delta z}{6} \mathbf{M}_0 \quad , \quad \mathbf{M}_0 = \begin{bmatrix} 4 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 1 & 4 & 1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 4 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 1 & 4 \end{bmatrix} \tag{9.23}$$

$$\mathbf{K} = \left[c \int_{z_{i-1}}^{z_{i+1}} N_j(z) \frac{d}{dz} N_i(z) dz + \frac{d}{2} \int_{z_{i-1}}^{z_{i+1}} N_j(z) \frac{d^2}{dz^2} N_i(z) dz \right] = \frac{c}{2} \mathbf{K}_0(Pe) \tag{9.24}$$

$$\mathbf{K}_0(Pe) = \begin{bmatrix} -\frac{2}{Pe} & -1 + \frac{1}{Pe} & 0 & \ldots & 0 & 0 & 0 \\ 1 + \frac{1}{Pe} & -\frac{2}{Pe} & -1 + \frac{1}{Pe} & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 + \frac{1}{Pe} & -\frac{2}{Pe} & -1 + \frac{1}{Pe} \\ 0 & 0 & 0 & \ldots & 0 & 1 + \frac{1}{Pe} & -\frac{2}{Pe} \end{bmatrix} \tag{9.25}$$

$$Pe = \frac{c\Delta z}{D} \tag{9.26}$$

where $\mathbf{M}$ and $\mathbf{K}$ are matrices with components M_{ij} and K_{ij}, and Pe is the socalled *Peclet number*. As seen, even in the case of standard Galerkin method $\mathbf{K}$ fails to be symmetric for $Pe > 0$. Only in the limit $Pe \to 0$ a symmetric matrix $\mathbf{K}$ is obtained, whereas the other extremes $Pe \to \pm\infty$ produce anti-symmetric matrices. At the evaluation of (9.25) the formal representation of $\frac{d^2}{dz^2} N_i(z)$ in terms of Dirac's delta-spikes as shown in fig. 9.1c have been used. (9.12) can then be written

$$\mathbf{M}_0 \frac{d}{d\tau} \mathbf{f}(\tau) - 3C\mathbf{K}_0(Pe)\mathbf{f}(\tau) = 0 \tag{9.27}$$

$$\tau = \frac{t}{\Delta t} \tag{9.28}$$

$$C = \frac{c\Delta t}{\Delta z} \tag{9.29}$$

In (9.28) time has been made non-dimensional with respect a the physical time step Δt used in the numerical integration of (9.12). C is the socalled *Courant number*. Pe

and C may be considered as non-dimensional parameters introduced in the formulation (9.27) instead of c and d. The Peclet number Pe is a measure of the magnitude of the convection term in proportion to the diffusion term in the Fokker-Planck equation, whereas the Courant number measures the convection during a time step relative to the mesh width. Further, due to the normalization (9.28), the time step has been fixed to $\Delta\tau = 1$ in all numerical schemes applied at the integration of (9.27).

Example 9.1: Properties of the standard Galerkin variational method

The case $N = 11$, $a = 0$, $b = 1$ is considered. Further the following uniformly distributed initial values are assumed $f_0(z) = 1.0$. The analytical solution of the problem can then be written

$$f(z,\tau) = \exp\Big((N+1)Pe \cdot z\Big) \sum_{j=1}^{\infty} F_j \exp(-\Lambda_j \tau) \sin(j\pi z) \tag{9.30}$$

$$\Lambda_j = \frac{1}{2}\frac{C}{Pe}\left(\left(\frac{j\pi}{N+1}\right)^2 + Pe^2\right) \tag{9.31}$$

$$F_j = 2\int_0^1 \exp\Big(-(N+1)Pe \cdot z\Big) \sin(j\pi z) dz = \frac{2j\pi}{(N+1)^2 Pe^2 + (j\pi)^2}\left(1 - (-1)^j e^{-(N+1)Pe}\right) \tag{9.32}$$

(9.30) is obtained by the standard technique of separation of variables. The rate of convergence of the series depends heavily on Pe and is generally very slow. In order to obtain 3 accurate digits 2000 terms are necessary for $Pe = 0.1$ and 300000 terms for $Pe = 1.0$.

If the integration of (9.27) is performed by means of a 4th order Runge-Kutta, instantaneous numerical instability is observed for all combinations of the Peclet and Courant numbers considered below. Instead the following unconditionally stable and second order accurate Crank-Nicholson scheme is applied

$$\left.\begin{aligned} &\Big(\mathbf{M}_0 - \tfrac{3}{2}C\mathbf{K}_0\Delta\tau\Big)\mathbf{f}(\tau+\Delta\tau) = \Big(\mathbf{M}_0 + \tfrac{3}{2}C\mathbf{K}_0\Delta\tau\Big)\mathbf{f}(\tau) \quad , \quad \Delta\tau = 1 \\ &\mathbf{f}(0) = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \end{aligned}\right\} \tag{9.33}$$

Fig. 9.2 shows the time-dependence of the solution at the upper quarter point $z = 0.75$, $f(0.75,\tau)$, as a function of the Courant and Peclet numbers. The parameter values $Pe = 0.1,\ 1,\ 10,\ 100$ and $C = 0.1,\ 1,\ 10$ are considered. Analytical solutions have only been obtained for $Pe = 0.1$ and $Pe = 1$ and are shown as an unbroken line in figs. 9.2a and 9.2b. As seen the agreement between analytical and numerical results is best at small Peclet and Courant numbers. For $C = 10$ rapid oscillatory solutions appear even in figs 9.2a and 9.2b. Figs. 9.2c and 9.2d show the results for the relatively high Peclet numbers of $Pe = 10$ and $Pe = 100$. In these cases large low-frequency oscillations occur in the numerical solutions, and the results should be considered meaningless. For c and d fixed the results in figs. 9.2c and 9.2d indicate that both the grid width Δz and the physical time step Δt must be reduced in order to reduce Pe and C. The overall conclusion drawn from this example then seems to be that qualitatively and quantitatively correct results based on the standard Galerkin approach in combination with some numerical time integrator, put severe limitations on the maximum values of the Peclet and Courant numbers, which should be checked locally throughout the solution domain.

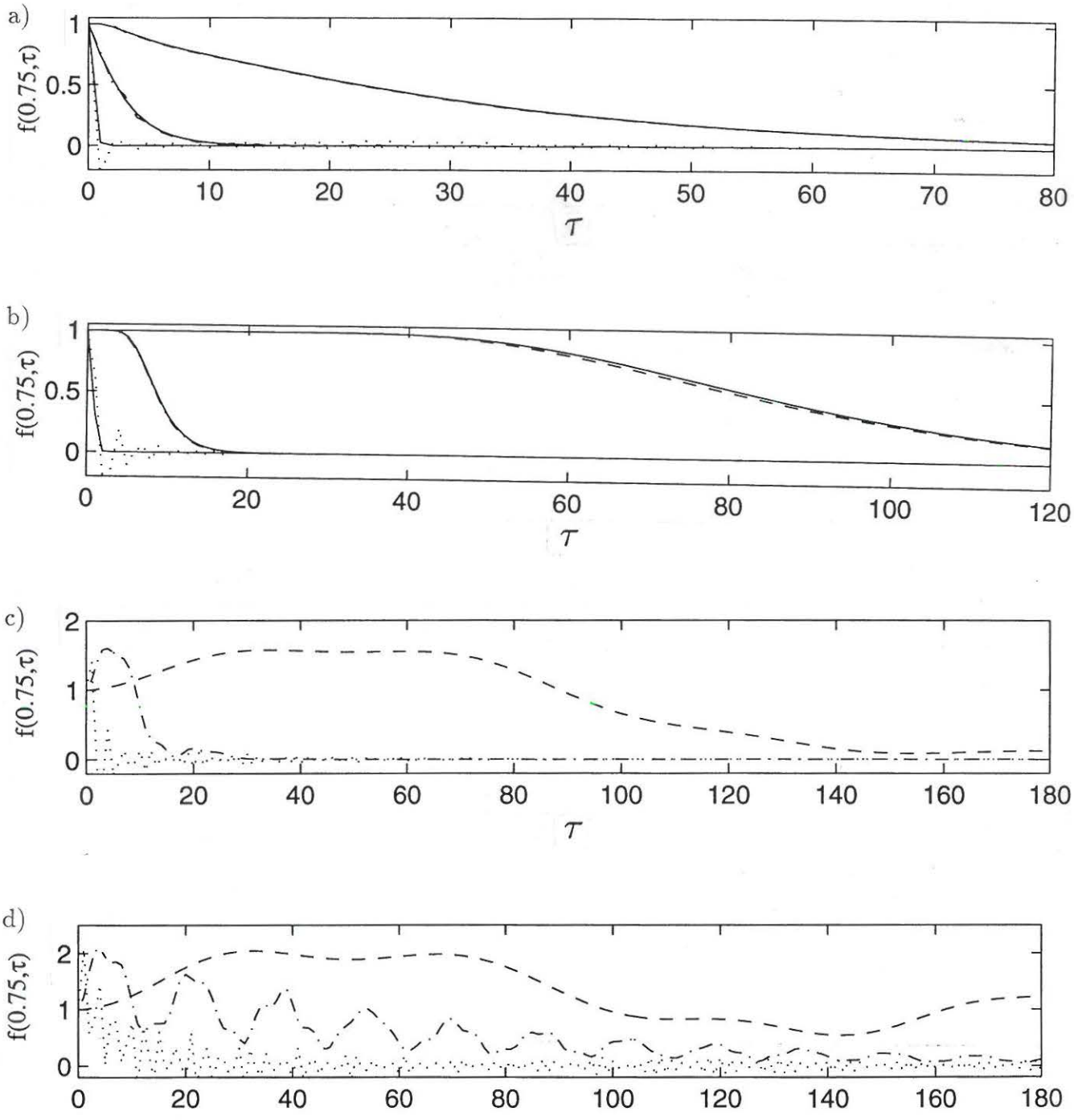

Fig. 9.2: Standard Galerkin variational method. Time-dependence of solution of the one-dimensional Fokker-Planck equation at the upper quarter point, $f(0.75, \tau)$, as a function of the Courant and Peclet numbers. $N = 11$, $a = 0$, $b = 1$. a) $Pe = 0.1$. b) $Pe = 1$. c) $Pe = 10$. d) $Pe = 100$. Analytical solutions: — . Numerical solutions: $C = 0.1$: - - - - , $C = 1$: -.-.-.- , $C = 10$: .... .

Based on the conclusions of example 9.1 the reasons for the malfunction of the standard Galerkin approach, when applied to multi-dimensional forward or backward integro-differential Chapman-Kolmogorov equations should now be obvious. No matter how small the time step and the grid width are selected the local Peclet number and the local Courant number will in some part of the solution space exceed acceptable limits, and numerically unstable or inaccurate results are obtained. In section 9.2 it is demonstrated, how this problem can be partially cured by using another choice of the weighting functions $V_i(\mathbf{z})$ than the standard Galerkin choice, $V_i(\mathbf{z}) = N_i(\mathbf{z})$. Further, applications to the first-passage time problem of a non-linear SDOF oscillator exposed to Gaussian white noise is demonstrated. In section 9.3 the method is modified for application to the same problem, when the oscillator is driven by a compound Poisson process.

9.1 Solution of the Fokker-Planck equation and the backward Kolmogorov equation

Initially, the solutions of initial-boundary value problem (9.1) by means of the finite difference method will be considered. Again, the drift and difffusion functions are assumed to be constants as functions of z and t. The following central finite difference approximations of local truncation error $O(\Delta z^2)$ may then be used for the partial derivatives with respect to z

$$\frac{\partial}{\partial z} f(z_j, t) \simeq \frac{f_{j+1}(t) - f_{j-1}(t)}{2\Delta z} \tag{9.34}$$

$$\frac{\partial^2}{\partial z^2} f(z_j, t) \simeq \frac{f_{j+1}(t) - 2f_j(t) + f_{j-1}(t)}{\Delta z^2} \tag{9.35}$$

where $f_j(t) = f(z_j, t)$, $z_j = j\Delta z$. As will be shown below (9.34), (9.35) lead to the differential system (9.27) with $\mathbf{K}_0(Pe)$ unchanged given by (9.25), whereas $\mathbf{M}_0$ is now given by the main diagonal (lumped "mass" tensor) defined below in eq. (9.40). The finite difference solution based on (9.34), (9.35) then results in the same poor solutions as the standard Galerkin method. Motivated by the fact that propagation of information is carried in the flow direction as indicated by the sign of c, an improvement was made in fluid mechanics at the attempts to solve the vorticity transport equation at large Reynolds numbers, a problem of the same type as the 2-dimensional Fokker-Planck equation, upon replacing the difference operator (9.34) by the following socalled *upwind difference* approximations

$$\frac{\partial}{\partial z} f(z_j, t) \simeq \begin{cases} \frac{f_j(t) - f_{j-1}(t)}{\Delta z} & , \quad c > 0 \\ \frac{f_{j+1}(t) - f_j(t)}{\Delta z} & , \quad c < 0 \end{cases} \tag{9.36}$$

Since (9.36) only has the local truncation error $O(\Delta z)$ this may at first sight appear as a bad idea. Nevertheless (9.36) turns out to produce qualitatively much better results at large Peclet numbers than (9.34). Finally, (9.34) and (9.36) can be combined into the following linear combination

$$\frac{\partial}{\partial z}f(z_j,t) \simeq \begin{cases} (1-\alpha_0)\frac{f_{j+1}(t)-f_{j-1}(t)}{2\Delta z} + \alpha_0 \frac{f_j(t)-f_{j-1}(t)}{\Delta z} & , \quad c>0 \\ (1-\alpha_0)\frac{f_{j+1}(t)-f_{j-1}(t)}{2\Delta z} + \alpha_0 \frac{f_{j+1}(t)-f_j(t)}{\Delta z} & , \quad c<0 \end{cases} \tag{9.37}$$

where α_0 is an arbitrary non-negative number. For $\alpha_0 = 0$ and $\alpha_0 = 1$ (9.37) reduce to (9.34) and (9.36), respectively. As for (9.36) the approximation (9.37) has the local truncation error $O(\Delta z)$, unless $\alpha_0 = 0$. Inserting (9.37) into the differential equation in (9.1) leads to the following system of ordinary differential equations

$$\begin{bmatrix} \dot{f}_1(t) \\ \dot{f}_2(t) \\ \vdots \\ \dot{f}_{n-1}(t) \\ \dot{f}_n(t) \end{bmatrix} = -\frac{c}{2\Delta z} \begin{bmatrix} 2\alpha & 1-\alpha & 0 & \dots & 0 & 0 & 0 \\ -1-\alpha & 2\alpha & 1-\alpha & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1-\alpha & 2\alpha & 1-\alpha \\ 0 & 0 & 0 & \dots & 0 & -1-\alpha & 2\alpha \end{bmatrix} \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_{n-1}(t) \\ f_n(t) \end{bmatrix}$$

$$+\frac{d}{2\Delta z^2} \begin{bmatrix} -2 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & -2 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -2 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_{n-1}(t) \\ f_n(t) \end{bmatrix} \tag{9.38}$$

where

$$\alpha = \begin{cases} \alpha_0 & , \quad c>0 \\ -\alpha_0 & , \quad c<0 \end{cases} \tag{9.39}$$

Using (9.26), (9.28), (9.29), eq. (9-38) can be written in the form (9.27) with $\mathbf{M}_0$ and $\mathbf{K}_0(Pe)$ given as

$$\mathbf{M}_0 = \begin{bmatrix} 6 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 6 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 6 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 6 \end{bmatrix} \tag{9.40}$$

$\mathbf{K}_0(Pe,\alpha) =$

$$\begin{bmatrix} -2\alpha - \frac{2}{Pe} & -1+\alpha+\frac{1}{Pe} & 0 & \dots & 0 & 0 & 0 \\ 1+\alpha+\frac{1}{Pe} & -2\alpha-\frac{2}{Pe} & -1+\alpha+\frac{1}{Pe} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1+\alpha+\frac{1}{Pe} & -2\alpha-\frac{2}{Pe} & -1+\alpha+\frac{1}{Pe} \\ 0 & 0 & 0 & \dots & 0 & 1+\alpha+\frac{1}{Pe} & -2\alpha-\frac{2}{Pe} \end{bmatrix} \tag{9.41}$$

As seen (9.41) reduces to (9.25) for $\alpha = 0$. Hence, the standard Galerkin method and the finite difference method with (9.34) and (9.35) are identical as previously stated. Since the finite difference method with the partial upwind differencing (9.37) turns out to favourable, one is motivated in trying to modify the variational method so that the solution for $\mathbf{K}_0(Pe, \alpha)$ as given by (9.41) is obtained also for $\alpha \neq 0$. This is performed by changing the weighting functions $V_i(z)$ slightly. The variational approach based on these modified weighting functions is known as the *Petrov-Galerkin variational method.* The weighting functions are taken on the form

$$V_i(z) = N_i(z) + \alpha W_i(z) \tag{9.42}$$

$$W_i(z) = -3\xi_i(1 - |\xi_i|) \quad , \quad \xi_i = \begin{cases} \frac{z - z_i}{\Delta z} & , \ z \in [z_{i-1}, z_{i+1}] \\ 0 & , \ z \notin [z_{i-1}, z_{i+1}] \end{cases} \tag{9.43}$$

$N_i(z)$ signifies the previous shape functions as shown in fig. 9.1. The *upwind differencing function* $W_i(z)$ as well as its first and second derivatives are shown in fig. 9.3.

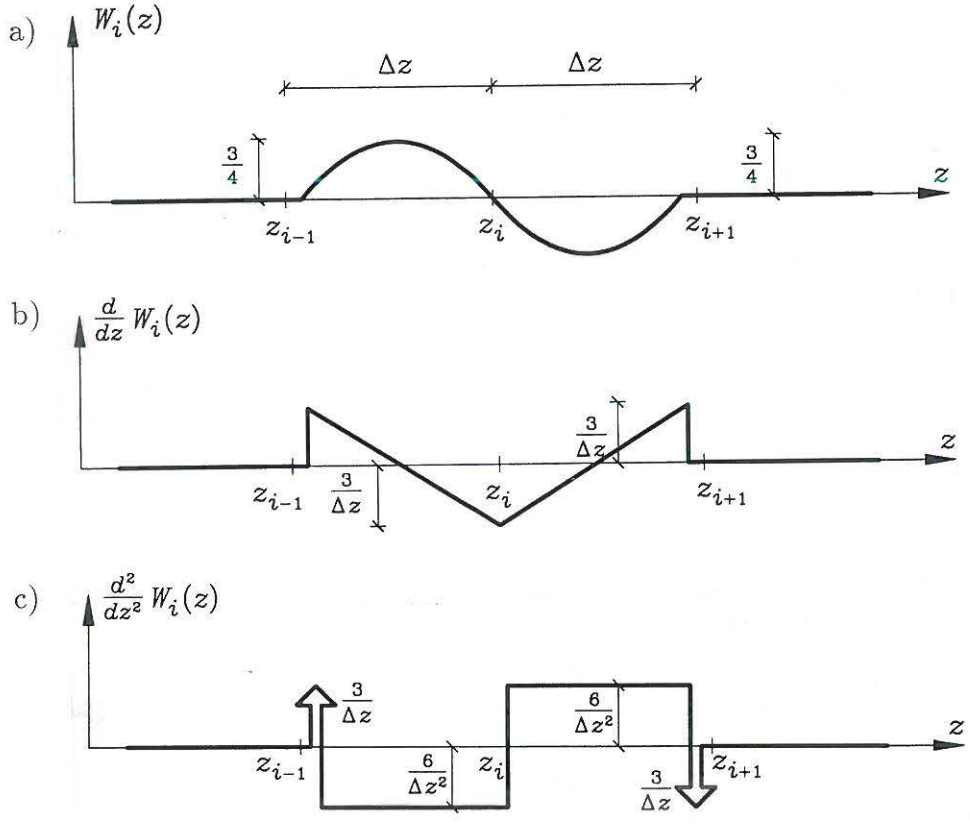

Fig. 9.3: Upwind differencing functions for one-dimensional Fokker-Planck equation. a) $W_i(z)$. b) $\frac{d}{dz} W_i(z)$. c) $\frac{d^2}{dz^2} W_i(z)$.

(9.16) and (9.17) become

$$\mathbf{M} = \left[\int_{z_{i-1}}^{z_{i+1}} V_i(z) N_j(z) dz \right] = \frac{\Delta z}{6} \mathbf{M}_0(\alpha) \tag{9.44}$$

$$\mathbf{M}_0(\alpha) = \begin{bmatrix} 4 & 1-\frac{3}{2}\alpha & 0 & \dots & 0 & 0 & 0 \\ 1+\frac{3}{2}\alpha & 4 & 1-\frac{3}{2}\alpha & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1+\frac{3}{2}\alpha & 4 & 1-\frac{3}{2}\alpha \\ 0 & 0 & 0 & \dots & 0 & 1+\frac{3}{2}\alpha & 4 \end{bmatrix} \tag{9.45}$$

$$\mathbf{K} = \left[c \int_{z_{i-1}}^{z_{i+1}} N_j(z) \frac{d}{dz} V_i(z) dz + \frac{d}{2} \int_{z_{i-1}}^{z_{i+1}} N_j(z) \frac{d^2}{dz^2} V_i(z) dz \right] = \frac{c}{2} \mathbf{K}_0(Pe, \alpha) \tag{9.46}$$

where $\mathbf{K}_0(Pe, \alpha)$ is given by (9.41). Hence, the upwind differencing functions (9.43) provide the required result. However the "mass" tensor has changed, and is now different from both (9.23) and (9.40). The indicated upwind differencing function is not the only solution to the problem. Actually, any function $W_i(z)$ with support in $[z_{i-1}, z_{i+1}$ will do, which is anti-symmetric around $z = z_i$ and which fulfils the normalization condition

$$\int_{z_{i-1}}^{z_i} W_i(z) dz = \frac{\Delta z}{2} \tag{9.47}$$

Differencing functions of this type provide the same "mass" tensor $\mathbf{M}_0(\alpha)$ given by (9.44) and "stiffness" tensor $\mathbf{K}_0(Pe, \alpha)$ given by (9.41).

Still the problem of the optimal selection of α remains. In order to answer this problem the stationary solution to (9.1) is considered, but now with the boundary conditions $f(a,t) = 0$, $f(b,t) = 1$. As previously the drift and diffusion functions are assumed to be constant. Then the solutions become

$$f(z, \infty) = \frac{\exp\left(\frac{2c}{d}(z-a)\right) - 1}{\exp\left(\frac{2c}{d}(b-a)\right) - 1} \quad \Rightarrow$$

$$f_j = f(z_j, \infty) = f(a + j\Delta z, \infty) = \frac{\exp\left(2Pe \cdot j\right) - 1}{\exp\left(2Pe \cdot (N+1)\right) - 1} \tag{9.48}$$

The corresponding numerial solution is obtained from the following difference equation which results from (9.41)

$$\left(1 + \alpha + \frac{1}{Pe}\right) f_{j-1} - \left(2\alpha + \frac{2}{Pe}\right) f_j + \left(-1 + \alpha + \frac{1}{Pe}\right) f_{j+1} = 0 \tag{9.49}$$

Insertion of (9.48) into (9.49) shows that the difference equation will provide the exact solution (9.48) for the following solution for α, Christie et al. [9.1]

$$\alpha_{\text{opt}} = \coth(Pe) - \frac{1}{Pe} \tag{9.50}$$

It should be noticed that (9.50) is only optimal for the present stationary problem. For the corresponding non-stationary problem or for other stationary or non-stationary problems exact solutions are not obtained from the discrete system with this choice for α. Nevertheless, (9.50) is often used also in these problems, since (9.50) will always improve the stability of the solutions compared to the standard Galerkin method. Notice that $Pe \to \infty$ implies that $\alpha \to 1$. Hence the full upwind differencing corresponding to the finite difference approximation (9.36) is obtained in the limit.

Next a finite element (FEM) formulation of the Petrov-Galerkin method is given. In order to achieve this (9.46) is written on the form

$$\mathbf{K} = \left[c \int_{z_{i-1}}^{z_{i+1}} N_j(z) \frac{d}{dz} V_i(z) dz - \frac{d}{2} \int_{z_{i-1}}^{z_{i+1}} \frac{d}{dz} N_j(z) \frac{d}{dz} V_i(z) dz \right] \tag{9.51}$$

(9.51) is obtained by integration by parts and use of $\frac{d}{dz} V_i(z_{i-1}^-) = \frac{d}{dx} V_i(z_{i+1}^+) = 0$. In the original formulation (9.7) of the variational principle $N_j(z)$ needs only be piecewise continuous, whereas (9.51) requires this property even for $\frac{d}{dz} N_j(z)$. Since the integrands of (9.51) are free of δ-spikes at $z = z_i^+$, $z = z_i$ and $z = z_{i+1}^-$, which are present in $\frac{d^2}{dz^2} V_i(z)$, see figs. 9.1c and 9.1c, the intergrals can be divided into two, one extending over $[z_{i-1}, z_i]$ and one extending over $[z_i, z_{i-1}]$ without any problems. This is the basic property need for a FEM formulation of the present problem, and it was in anticipation of such an application that the continuous shape function $N_i(z)$ shown in fig. 9.1a and the continuous upwind differencing function $W_i(z)$ shown in fig. 9.3a were selected.

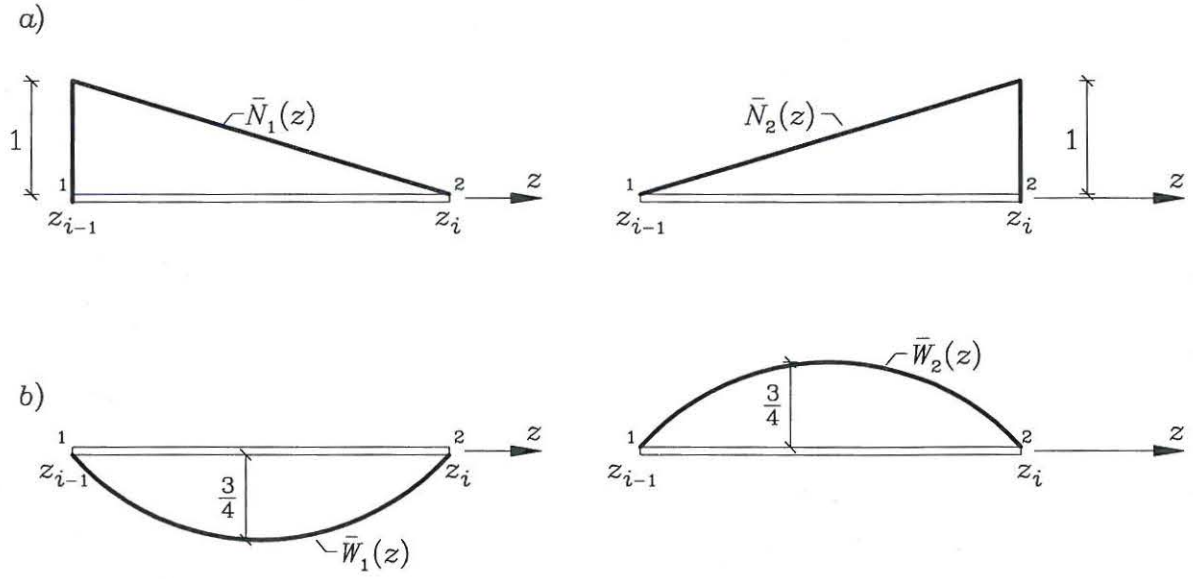

Fig. 9.4: a) Element shape functions. b) Element upwind differencing functions.

The interval $[a, b]$ is divided into $N+1$ elements, where the ith element extends over the interval $[z_{i-1}, z_i]$. The local nodes at the cooordinate z_{i-1} and z_i are denoted 1 and 2. The *element shape functions* $\overline{N}_1(z)$, $\overline{N}_2(z)$ and *element upwind differencing functions* $\overline{W}_1(z)$, $\overline{W}_2(z)$ are shown in fig. 9.4. These are parts of the corresponding global shape and upwind differencing functions in figs. 9.1a and 9.3a. The following element "mass" and "stiffness" matrices can then be calculated for element i

$$\mathbf{m}_i = \left[\int_{z_{i-1}}^{z_i} \overline{N}_j(z)\overline{V}_i(z)dz \right] = \frac{\Delta z}{6} \begin{bmatrix} 2-\frac{3}{2}\alpha & 1-\frac{3}{2}\alpha \\ 1+\frac{3}{2}\alpha & 2+\frac{3}{2}\alpha \end{bmatrix} \tag{9.52}$$

$$\mathbf{k}_i = \left[c \int_{z_{i-1}}^{z_i} \overline{N}_j(z)\frac{d}{dz}\overline{V}_i(z)dz - \frac{d}{2}\int_{z_{i-1}}^{z_i} \frac{d}{dz}\overline{N}_j(z)\frac{d}{dz}\overline{V}_i(z)dz \right] =$$

$$\frac{c}{2} \begin{bmatrix} -1-\alpha-\frac{1}{Pe} & -1+\alpha+\frac{1}{Pe} \\ 1+\alpha+\frac{1}{Pe} & 1-\alpha-\frac{1}{Pe} \end{bmatrix} \tag{9.53}$$

For c and d the mid-point values at the time t, $c\left(\frac{1}{2}(z_{i-1}+z_i), t\right)$, $d\left(\frac{1}{2}(z_{i-1}+z_i), t\right)$ are used. Pe and α should then be calculated according to these local numbers.

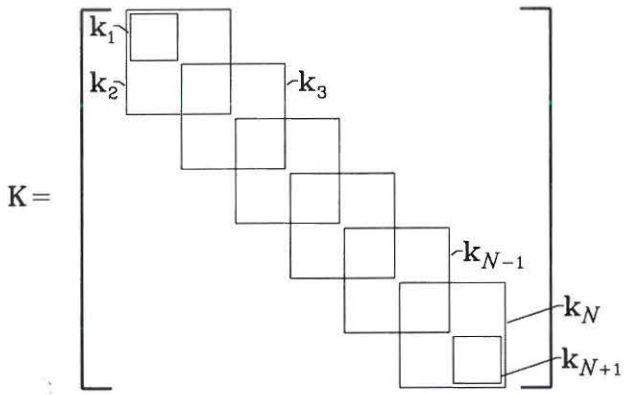

Fig. 9.5: Structure of global "stiffness" matrix for problem (9.1).

The global "stiffness" tensor is next assembled in the usual way by adding local stiffness matrices according to the topology of the network and corrected for the boundary conditions. The structure of the global stiffness matrix for the problem has been illustrated in fig. (9.5), which obviously results in the global stiffnessmatrix (9.46). A similar assembling and boundary value correction procedure is performed for the global "mass" matrix.

Example 9.2: Properties of the Petrov-Galerkin variational method

Below in fig. 9.6 the results for the same problem as considered in example 9.1 are shown. However, now upwind differencing with α calculated by (9.50) is applied. For $Pe = 0.1$ and $Pe = 1.0$ almost the same results as indicated in figs. 9.2a and 9.2b are obtained. The deviation between the analytical and numerical results for $Pe = 1$, $C = 0.1$ is slightly larger in fig. 9.6b than in fig. 9.2b. This is because the results of fig. 9.2b are based essentially on the central difference approximation (9.34) with the local truncation error $O(\Delta z^2)$, whereas the results of fig. 9.6b are based on (9.37) with the local truncation error $O(\Delta z)$. Comparing figs. 9.6c and 9.6d with figs. 9.2c and 9.2d the benefit of the upwind differencing is striking. Stable non-oscillatory solutions are now obtained for all cases of Pe considered, although the rapid oscillations at $C = 10$ are still present. These can only be removed upon reducing the physical time step. From this example it is then concluded that problems with large convection terms can be handled numerically, if upwind differencing is performed as is the case in the Petrov-Galerkin variational method.

Consider the non-linear and non-hysteretic SDOF system (1.86) exposed to Gaussian white noise. Formally the stochastic equation of motion can then be written

$$\ddot{Y} + g(Y, \dot{Y}) = \sqrt{d}\frac{dW(t)}{dt} \tag{9.54}$$

where $\{W(t), t \in [0, \infty[\}$ is a unit intensity Wiener process, see (1.7). The mass of the oscillator has been absorbed into the diffusion constant d. The drift- and diffusion vectors then become

$$\mathbf{c}(y, \dot{y}) = \begin{bmatrix} \dot{Y} \\ -g(y, \dot{y}Y) \end{bmatrix} \tag{9.55}$$

$$\mathbf{d} = \begin{bmatrix} 0 \\ D \end{bmatrix} \tag{9.56}$$

The solution domain is taken as $S = \{(y, \dot{y}) |]a, b[\times] - \infty, \infty[\}$. With $\mathbf{c}(y, \dot{y})$ and $\mathbf{d}$ given by (9.55) and (9.56) the initial- and boundary value problem becomes, cf. (2.73)

$$\left.\begin{array}{l} \frac{\partial}{\partial t} f(y, \dot{y}, t) = -\dot{y}\frac{\partial}{\partial y} f(y, \dot{y}, t) + \\ \frac{\partial}{\partial \dot{y}}\left(g(y, \dot{y}) f(y, \dot{y}, t)\right) + \frac{d}{2}\frac{\partial^2}{\partial \dot{y}^2} f(y, \dot{y}, t) \quad , \quad \forall\, t \in]0, \infty[\quad , \quad \forall\, (y, \dot{y}) \in S \\ f(y, \dot{y}, 0) = f_0(y, \dot{y}) \quad , \quad \forall\, (y, \dot{y}) \in S \\ f(y, \dot{y}, t) = 0 \quad , \quad \forall\, t \in]0, \infty[\quad , \quad \forall\, (y, \dot{y}) \in \partial S^{(0)} \cup \partial S^{(2)} \end{array}\right\} \tag{9.57}$$

(9.57) has been illustrated in fig. 9.7a. The interval S is divided into rectangular elements. N_1 elements are used in the y-directions and N_2 elements in the $\dot{y}$-direction. The elements all have the the same side lengths $\Delta y = \frac{b-a}{N_1}$ and $\Delta \dot{y}$. $\Delta \dot{y}$ should be selected so the boundary $\partial S^{(2)}$ at the ordinates $\dot{y} = \pm\frac{1}{2} N_2 \Delta \dot{y}$ will only be accessed with negligible probability. In fig. 9.7a the $N_1(N_2 - 1) + 1$ free system nodes are marked with a bullet.

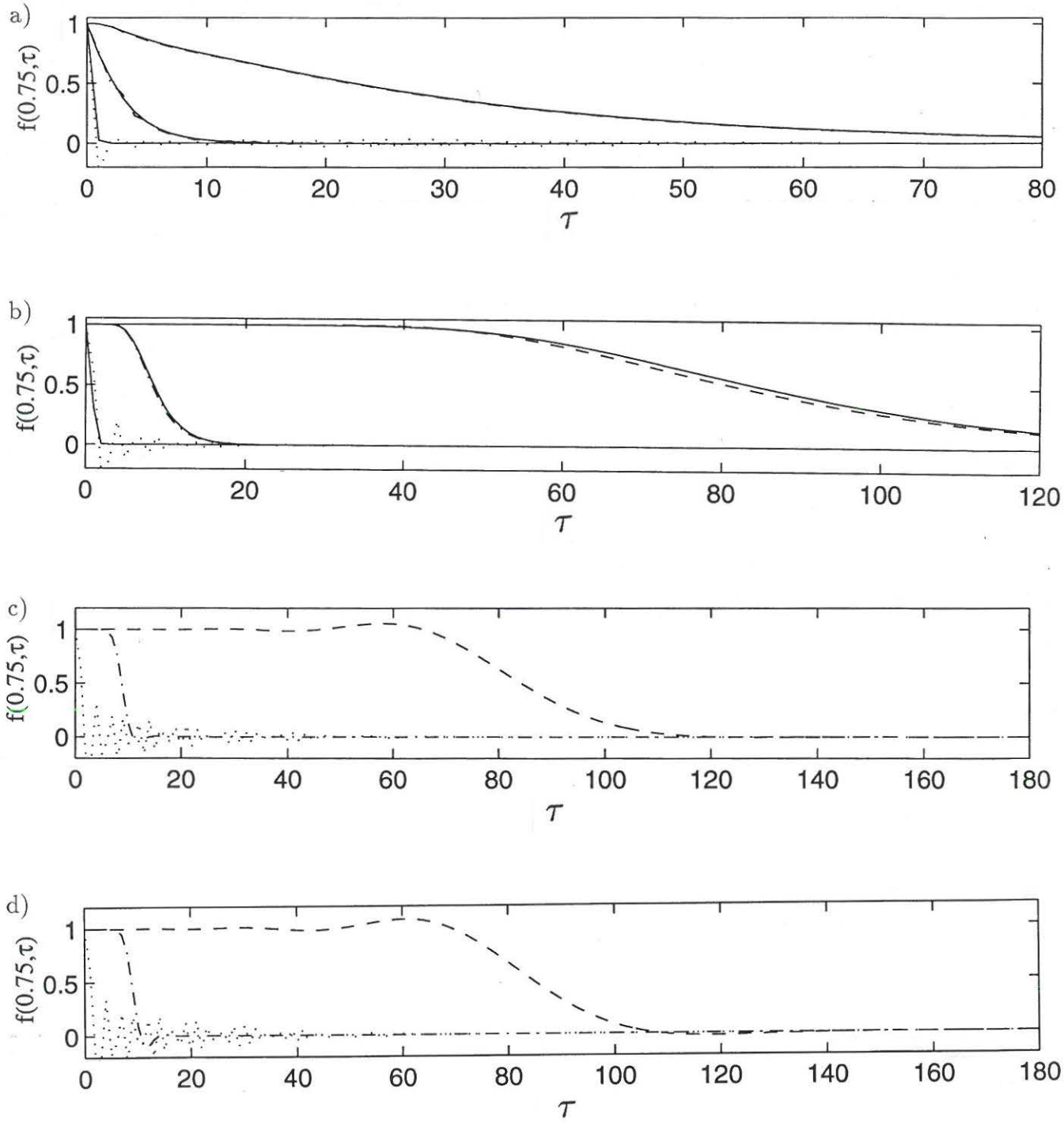

Fig. 9.6: Petrov-Galerkin variational method. Time-dependence of solution of the one-dimensional Fokker-Planck equation at the upper quarter point, $f(0.75, \tau)$, as a function of the Courant and Peclet numbers. $N = 11$, $a = 0$, $b = 1$. a) $Pe = 0.1$. b) $Pe = 1$. c) $Pe = 10$. d) $Pe = 100$. Analytical solutions: —- . Numerical solutions: $C = 0.1$: - - - - , $C = 1$: -.-.-.- , $C = 10$: .... .

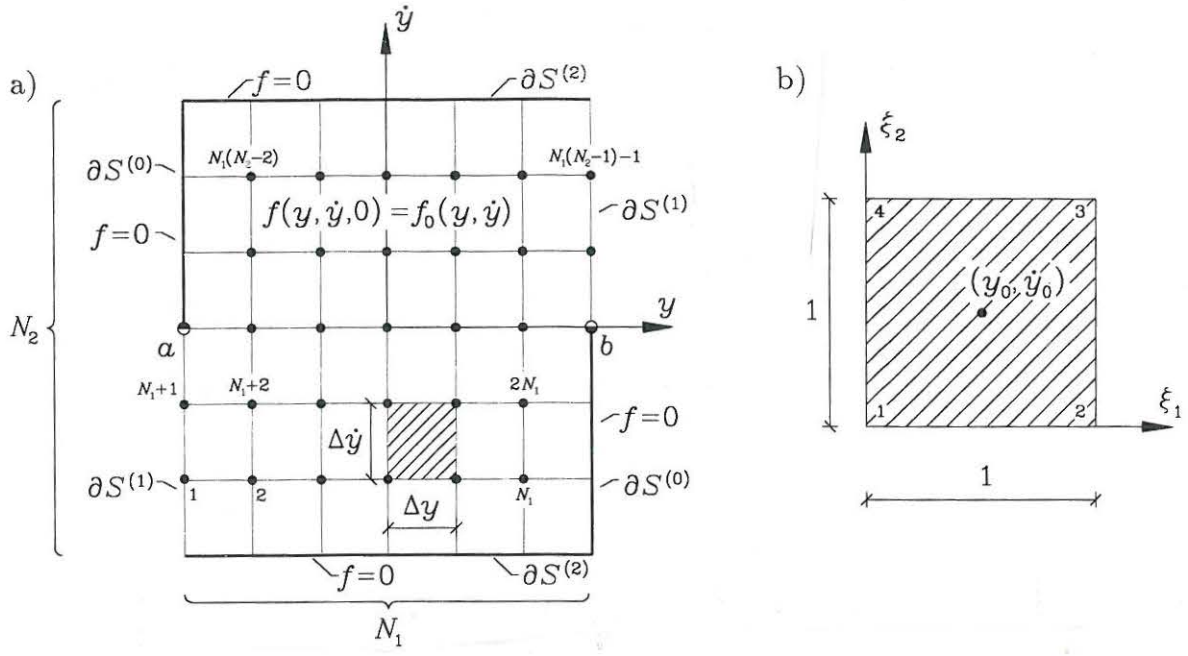

Fig. 9.7: Initial- and boundary value problem for two-dimensional Fokker-Planck equation of SDOF non-hysteretic oscillator. a) Element mesh and global element numbering. b) Local element coordinate system and element node numbering.

In fig. 7.9b the local element node numbering is defined. $(y_0, \dot{y}_0)$ signifies the global coordinates of the centroid of the element. Local non-dimensional coordinates (ξ_1, ξ_2) are introduced, which are related to the global coordinates $(y, \dot{y})$ by the transformation

$$\xi_1 = \frac{1}{2} + \frac{y - y_0}{\Delta y} \quad , \quad \xi_2 = \frac{1}{2} + \frac{\dot{y} - \dot{y}_0}{\Delta \dot{y}} \tag{9.58}$$

The following element shape and weighting functions, referring to the element nodes, can then be defined

$$\left.\begin{aligned}
N_1(y,\dot{y}) &= \overline{N}_1(\xi_1)\overline{N}_1(\xi_2) \quad , \quad V_1(y,\dot{y}) = \Big(\overline{N}_1(\xi_1) + \alpha_1 \overline{W}_1(\xi_1)\Big)\Big(\overline{N}_1(\xi_2) + \alpha_2 \overline{W}_1(\xi_2)\Big) \\
N_2(y,\dot{y}) &= \overline{N}_2(\xi_1)\overline{N}_1(\xi_2) \quad , \quad V_2(y,\dot{y}) = \Big(\overline{N}_2(\xi_1) + \alpha_1 \overline{W}_2(\xi_1)\Big)\Big(\overline{N}_1(\xi_2) + \alpha_2 \overline{W}_1(\xi_2)\Big) \\
N_3(y,\dot{y}) &= \overline{N}_2(\xi_1)\overline{N}_2(\xi_2) \quad , \quad V_3(y,\dot{y}) = \Big(\overline{N}_2(\xi_1) + \alpha_1 \overline{W}_2(\xi_1)\Big)\Big(\overline{N}_2(\xi_2) + \alpha_2 \overline{W}_2(\xi_2)\Big) \\
N_4(y,\dot{y}) &= \overline{N}_1(\xi_1)\overline{N}_2(\xi_2) \quad , \quad V_4(y,\dot{y}) = \Big(\overline{N}_1(\xi_1) + \alpha_1 \overline{W}_1(\xi_1)\Big)\Big(\overline{N}_2(\xi_2) + \alpha_2 \overline{W}_2(\xi_2)\Big)
\end{aligned}\right\} \tag{9.59}$$

where $\overline{N}_1(\xi)$, $\overline{N}_2(\xi)$, $\overline{W}_1(\xi)$, $\overline{W}_2(\xi)$ signify the one-dimensional shape functions and upwind differencing functions illustrated in fig. 9.4. As seen, different upwind differencings have been used in the y- and $\dot{y}$-directions. Convection but no diffusion occurs in the

y-direction (the diffusion coefficient as a factor to $\frac{\partial^2}{\partial y^2} f(y, \dot{y}, t)$ in (9.57) is 0). The convection components in the two directions follow from (9.55). The following local Peclet numbers in the two directions can then be defined. cf. (9.26)

$$Pe_1 = \frac{\dot{y}_0 \Delta y}{0} = \begin{cases} \infty & , \ \dot{y}_0 > 0 \\ -\infty & , \ \dot{y}_0 < 0 \end{cases} \tag{9.60}$$

$$Pe_2 = \frac{-g(y_0, \dot{y}_0)\Delta \dot{y}}{d} \tag{9.61}$$

The product form of the shape and weighting functions (9.59) as well as the assumptions on the directional convection and diffusion as follows from the settings (9.60) and (9.61) are due to Bergman and Heinrich [9.3]. Using the optimality criterion (9.50), the upwind differencing parameters α_1 and α_2 then become

$$\alpha_1 = \begin{cases} 1 & , \ \dot{y}_0 > 0 \\ -1 & , \ \dot{y}_0 < 0 \end{cases} \tag{9.62}$$

$$\alpha_2 = \coth(Pe_2) - \frac{1}{Pe_2} \tag{9.63}$$

(9.62) implies that full upwind differencing is assumed in the y-directions. The element "mass" and "stiffness" matrices $\mathbf{m}_{\mathrm{FP}}$ and $\mathbf{m}_{\mathrm{FP}}$ for the two-dimensional Fokker-Planck equation now become, cf. (2.74), (9.10), (9.11)

$$\mathbf{m}_{\mathrm{FP}} = \left[m_{ij}\right] = \Delta y \Delta \dot{y} \left[\int_0^1 \int_0^1 N_j(y, \dot{y}) V_i(y, \dot{y}) d\xi_1 d\xi_2 \right] = \frac{\Delta y \Delta \dot{y}}{36} \mathbf{m}_0 \tag{9.64}$$

$$\mathbf{m}_0 = \begin{bmatrix} (2-\frac{3}{2}\alpha_1)(2-\frac{3}{2}\alpha_2) & (1-\frac{3}{2}\alpha_1)(2-\frac{3}{2}\alpha_2) & (1-\frac{3}{2}\alpha_1)(1-\frac{3}{2}\alpha_2) & (2-\frac{3}{2}\alpha_1)(1-\frac{3}{2}\alpha_2) \\ (1+\frac{3}{2}\alpha_1)(2-\frac{3}{2}\alpha_2) & (2+\frac{3}{2}\alpha_1)(2-\frac{3}{2}\alpha_2) & (2+\frac{3}{2}\alpha_1)(1-\frac{3}{2}\alpha_2) & (1+\frac{3}{2}\alpha_1)(1-\frac{3}{2}\alpha_2) \\ (1+\frac{3}{2}\alpha_1)(1+\frac{3}{2}\alpha_2) & (2+\frac{3}{2}\alpha_1)(1+\frac{3}{2}\alpha_2) & (2+\frac{3}{2}\alpha_1)(2+\frac{3}{2}\alpha_2) & (1+\frac{3}{2}\alpha_1)(2+\frac{3}{2}\alpha_2) \\ (2-\frac{3}{2}\alpha_1)(1+\frac{3}{2}\alpha_2) & (1-\frac{3}{2}\alpha_1)(1+\frac{3}{2}\alpha_2) & (1-\frac{3}{2}\alpha_1)(2+\frac{3}{2}\alpha_2) & (2-\frac{3}{2}\alpha_1)(2+\frac{3}{2}\alpha_2) \end{bmatrix} \tag{9.65}$$

$$\mathbf{k}_{\mathrm{FP}} = \left[k_{ij}\right] = \Delta y \Delta \dot{y} \left[\int_0^1 \int_0^1 N_j(y, \dot{y}) \mathcal{K}_{\mathbf{z}}^T [V_i(y, \dot{y})] d\xi_1 d\xi_2 \right] =$$

$$\left[\Delta y \Delta \dot{y} \int_0^1 \int_0^1 N_j(y, \dot{y}) \Big(\dot{y} \frac{\partial}{\partial y} V_i(y, \dot{y}) - g(y, \dot{y}) \frac{\partial}{\partial \dot{y}} V_i(y, \dot{y}) + \frac{d}{2} \frac{\partial^2}{\partial \dot{y}^2} V_i(y, \dot{y}) \Big) d\xi_1 d\xi_2 \right] =$$

$$\Big[\Delta y\Delta\dot{y}\int_0^1\int_0^1 N_j(y,\dot{y})\Big(\dot{y}\frac{\partial}{\partial y}V_i(y,\dot{y})-g(y,\dot{y})\frac{\partial}{\partial\dot{y}}V_i(y,\dot{y})\Big)d\xi_1 d\xi_2\ -$$

$$\Delta y\Delta\dot{y}\int_0^1\int_0^1 \frac{d}{2}\frac{\partial}{\partial\dot{y}}N_j(y,\dot{y})\frac{\partial}{\partial\dot{y}}V_i(y,\dot{y})d\xi_1 d\xi_2\Big]=\dot{y}_0\frac{\Delta\dot{y}}{12}\mathbf{k}_1-g(y_0,\dot{y}_0)\frac{\Delta y}{12}\mathbf{k}_2-d\frac{\Delta y}{12\Delta\dot{y}}\mathbf{k}_3 \tag{9.66}$$

$$\mathbf{k}_1=\begin{bmatrix} -(1+\alpha_1)(2-\frac{3}{2}\alpha_2) & -(1-\alpha_1)(2-\frac{3}{2}\alpha_2) & -(1-\alpha_1)(1-\frac{3}{2}\alpha_2) & -(1+\alpha_1)(1-\frac{3}{2}\alpha_2)\\ (1+\alpha_1)(2-\frac{3}{2}\alpha_2) & (1-\alpha_1)(2-\frac{3}{2}\alpha_2) & (1-\alpha_1)(1-\frac{3}{2}\alpha_2) & (1+\alpha_1)(1-\frac{3}{2}\alpha_2)\\ (1+\alpha_1)(1+\frac{3}{2}\alpha_2) & (1-\alpha_1)(1+\frac{3}{2}\alpha_2) & (1-\alpha_1)(2+\frac{3}{2}\alpha_2) & (1+\alpha_1)(2+\frac{3}{2}\alpha_2)\\ -(1+\alpha_1)(1+\frac{3}{2}\alpha_2) & -(1-\alpha_1)(1+\frac{3}{2}\alpha_2) & -(1-\alpha_1)(2+\frac{3}{2}\alpha_2) & -(1+\alpha_1)(2+\frac{3}{2}\alpha_2)\end{bmatrix} \tag{9.67}$$

$$\mathbf{k}_2=\begin{bmatrix} -(2-\frac{3}{2}\alpha_1)(1+\alpha_2) & -(1-\frac{3}{2}\alpha_1)(1+\alpha_2) & -(1-\frac{3}{2}\alpha_1)(1-\alpha_2) & -(2-\frac{3}{2}\alpha_1)(1-\alpha_2)\\ -(1+\frac{3}{2}\alpha_1)(1+\alpha_2) & -(2+\frac{3}{2}\alpha_1)(1+\alpha_2) & -(2+\frac{3}{2}\alpha_1)(1-\alpha_2) & -(1+\frac{3}{2}\alpha_1)(1-\alpha_2)\\ (1+\frac{3}{2}\alpha_1)(1+\alpha_2) & (2+\frac{3}{2}\alpha_1)(1+\alpha_2) & (2+\frac{3}{2}\alpha_1)(1-\alpha_2) & (1+\frac{3}{2}\alpha_1)(1-\alpha_2)\\ (2-\frac{3}{2}\alpha_1)(1+\alpha_2) & (1-\frac{3}{2}\alpha_1)(1+\alpha_2) & (1-\frac{3}{2}\alpha_1)(1-\alpha_2) & (2-\frac{3}{2}\alpha_1)(1-\alpha_2)\end{bmatrix} \tag{9.68}$$

$$\mathbf{k}_3=\begin{bmatrix} (2-\frac{3}{2}\alpha_1) & (1-\frac{3}{2}\alpha_1) & -(1-\frac{3}{2}\alpha_1) & -(2-\frac{3}{2}\alpha_1)\\ (1+\frac{3}{2}\alpha_1) & (2+\frac{3}{2}\alpha_1) & -(2+\frac{3}{2}\alpha_1) & -(1+\frac{3}{2}\alpha_1)\\ -(1+\frac{3}{2}\alpha_1) & -(2+\frac{3}{2}\alpha_1) & (2+\frac{3}{2}\alpha_1) & (1+\frac{3}{2}\alpha_1)\\ -(2-\frac{3}{2}\alpha_1) & -(1-\frac{3}{2}\alpha_1) & (1-\frac{3}{2}\alpha_1) & (2-\frac{3}{2}\alpha_1)\end{bmatrix} \tag{9.69}$$

As in (9.51) integration by parts has been performed over two adjacent elements to remove the second derivative with respect to $\dot{y}$ in the backward Kolmogorov operator, and hence to insure piecewise continuous functions, so the integrals can be separated at the element boundary. At the evaluation of the integrals in (9.66) $\dot{y}$ and $g(y,\dot{y})$

have been considered constantly equal to the values in the centroid. Of course a better evaluation can be obtained if an iso-parametric interpolation of these functions between their nodal values is performed.

Next, the local "mass" and "stiffness" matrices are assembled into the global "mass" and "stiffness" matrices $\mathbf{M}$ and $\mathbf{K}$ of dimension $\big(N_1(N_2-1)-1\big)\times\big(N_1(N_2-1)-1\big)$ according to the topology of the network, which follows from the node numbering in fig. 9.7a. Let $\mathbf{M}_0$, $\mathbf{K}_1$, $\mathbf{K}_2$, and $\mathbf{K}_3$ signify the assemblages by the finite element methods of the element matrices $\mathbf{m}_0$, $\mathbf{k}_1$, $\mathbf{k}_2$ and $\mathbf{k}_3$ as given by (9.65), (9.67), (9.68) and (9.69). A double indexing of the nodal points is introduced. Node (i,j) in the mesh signifies the ith node in the y-direction and the jth node in the $\dot{y}$-direction. The coupling between the node (i,j) and its surrounding nodes, as follows from the corresponding row in $\mathbf{M}_0$, $\mathbf{K}_1$, $\mathbf{K}_2$ and $\mathbf{K}_3$ can next be shown to be equivalent to the finite difference stencils depicted in figs. 9.8, 9.9, 9.10 and 9.11, respectively. An analysis of the finite difference scheme in fig. 9.8 based on a Taylor expansions of the nodal values from the central node reveals that the scheme is equal to $\Delta y\Delta\dot{y}\big[36\frac{\partial f_{ij}}{\partial t}+\mathrm{O}(\max(\Delta y,\Delta\dot{y}))\big]$, where f_{ij} signifies the solution for node (i,j). Similarly, the finite difference schemes in figs. 9.9, 9.10 and 9.11 are equal to the values $\Delta y\big[-12\frac{\partial f_{ij}}{\partial y}+\mathrm{O}(\max(\Delta y,\Delta\dot{y}))\big]$, $\Delta\dot{y}\big[-12\frac{\partial f_{ij}}{\partial\dot{y}}+\mathrm{O}(\max(\Delta y,\Delta\dot{y}))\big]$ and $\Delta\dot{y}^2\big[-6\frac{\partial^2 f_{ij}}{\partial\dot{y}^2}+\mathrm{O}(\max(\Delta y,\Delta\dot{y}))\big]$. When these results are combined with the weights indicated in (9.64) and (9.66) it is seen that the system equation equations (9.12) represents the differential equation in (9.57) with a local truncation error of magnitude $\mathrm{O}(\max(\Delta y,\Delta\dot{y}))$. (9.12) will be solved numerically by means of the following Crank-Nicholson scheme, cf. (9.33)

$$\left.\begin{aligned}&\Big(\mathbf{M}-\tfrac{1}{2}\Delta t\mathbf{K}\Big)\mathbf{f}(t+\Delta t)=\Big(\mathbf{M}+\tfrac{1}{2}\Delta t\mathbf{K}\Big)\mathbf{f}(t)\\&\mathbf{f}(0)=\begin{bmatrix}f_0(y_1,\dot{y}_1)\\\vdots\\f_0(y_{N_1(N_2-1)-1},\dot{y}_{N_1(N_2-1)-1})\end{bmatrix}\end{aligned}\right\}\tag{9.70}$$

The initial-boundary value problem determines the probability mass within S as more and more probability mass is absorbed on the entrance boundary $\partial S^{(0)}$. However, if $-a$ and b are chosen sufficiently large compared to the stationary standard deviation $\sigma_Y(\infty)$, the absorption will be minimal, and the system of equations can be used to calculate the non-stationary stochastic response of the oscillator. However, from time to time the probability mass within S should be updated to 1 to correct for the small absorption taking place. The stationary distribution $\mathbf{f}(\infty)$ is seen to fulfil

$$\mathbf{f}(\infty)=\mathbf{A}\mathbf{f}(\infty)\quad,\quad\mathbf{A}=\Big(\mathbf{M}-\frac{1}{2}\Delta t\mathbf{K}\Big)^{-1}\Big(\mathbf{M}+\frac{1}{2}\Delta t\mathbf{K}\Big)\tag{9.71}$$

(9.71) determines $\mathbf{f}(\infty)$ as the normed eigenvector related to the eigenvalue $\lambda=1$ of the matrix $\mathbf{A}$. Since the eigenvalue is known, $\mathbf{f}(\infty)$ is determined more effectively from the eigenvalue problem (9.71), rather than iterating the map (9.70) until stationarity. $\lambda=1$ is the largest eigenvalue of $\mathbf{A}$. Then the latter approach is merely the well-known power method for solving an eigenvalue problem.

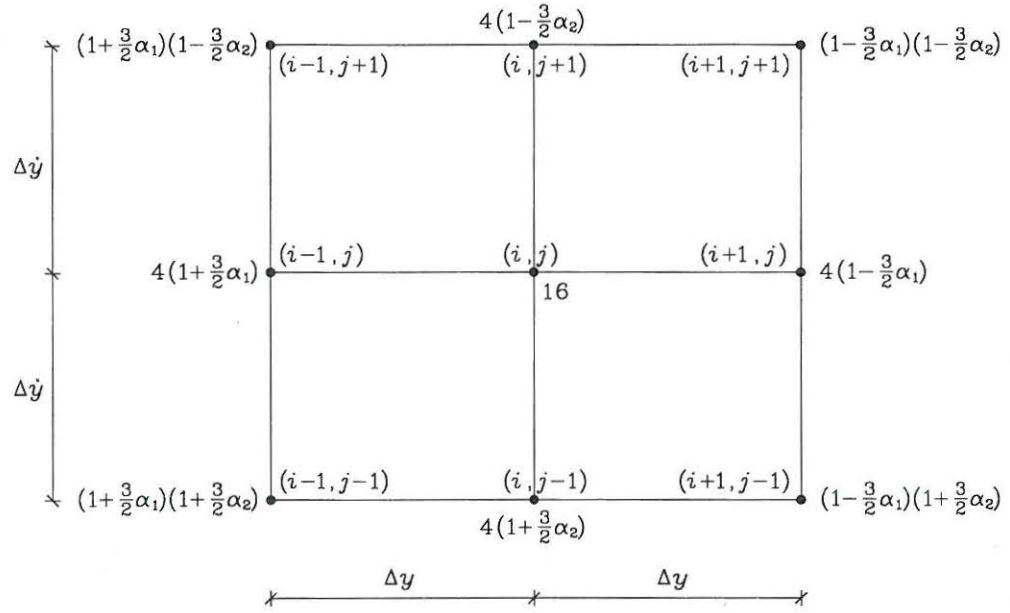

Fig. 9.8: Equivalent finite difference scheme for $\mathbf{M}_0\dot{\mathbf{f}}$.

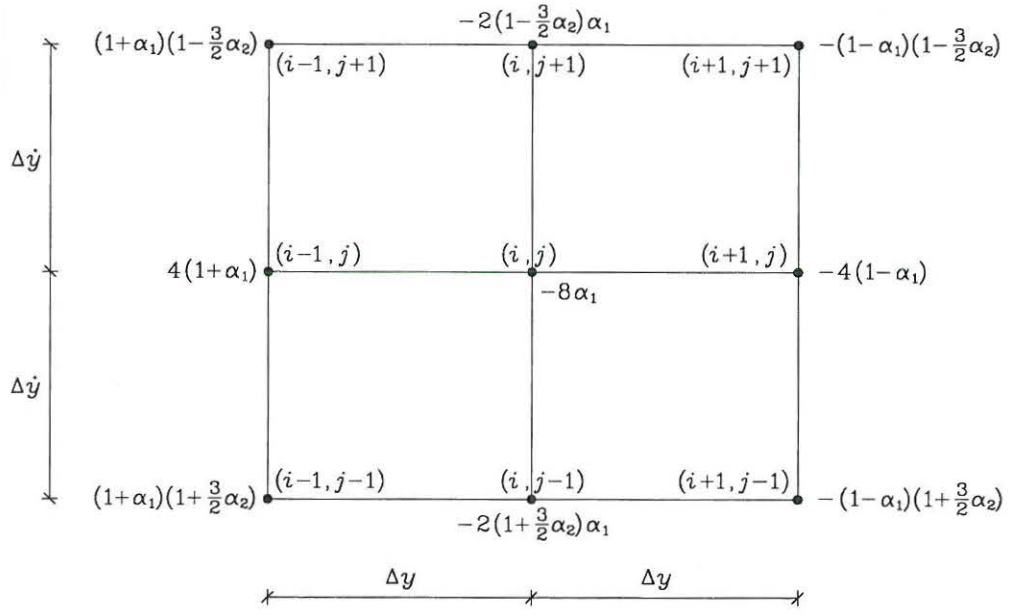

Fig. 9.9: Equivalent finite difference scheme for $\mathbf{K}_1\mathbf{f}$.

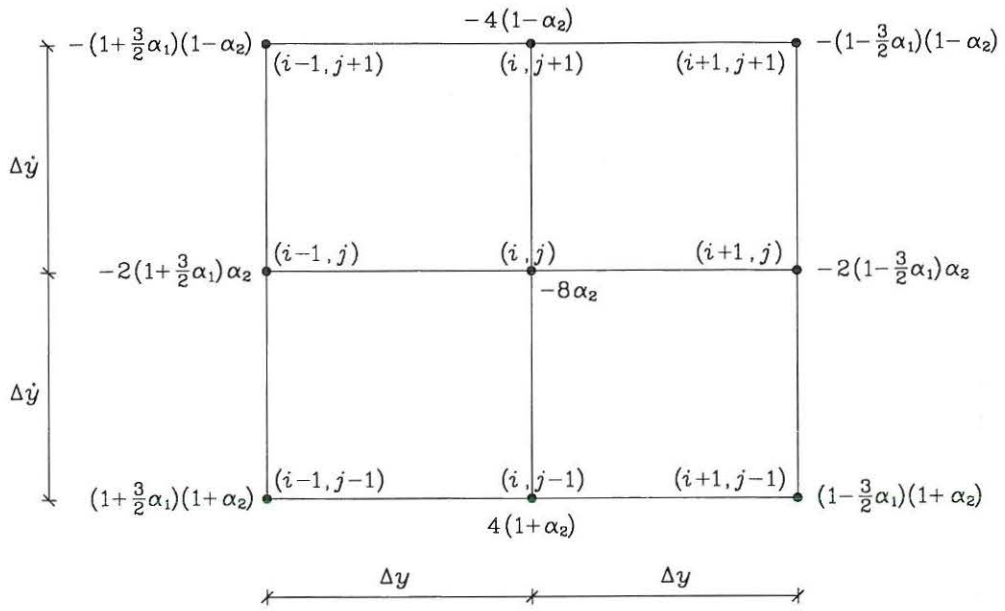

Fig. 9.10: Equivalent finite difference scheme for $\mathbf{K}_2\mathbf{f}$.

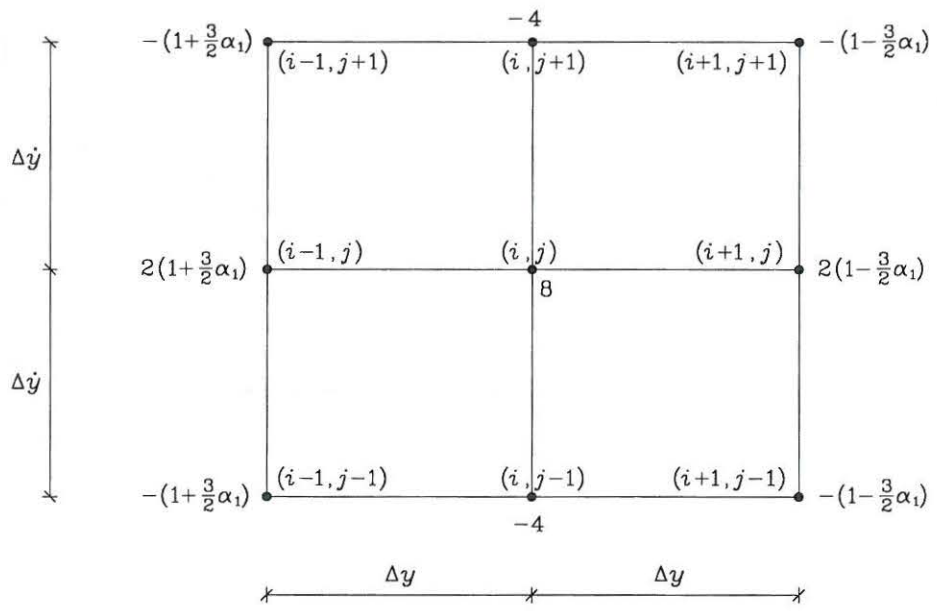

Fig. 9.11: Equivalent finite difference scheme for $\mathbf{K}_3\mathbf{f}$.

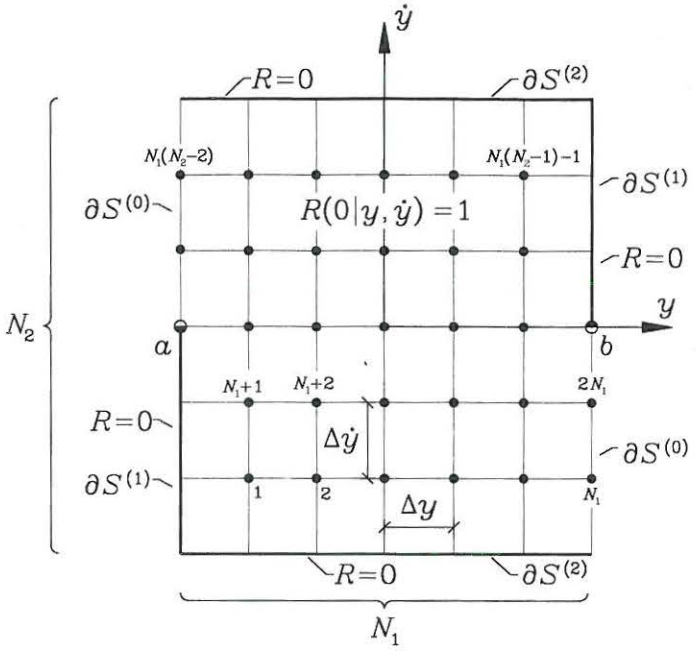

Fig. 9.12: Initial- and boundary value problem for the determination of the reliability function of SDOF non-hysteretic oscillator. Element mesh and global element numbering.

Next, the boundary and initial value problem (6.38) for the determination of the first-passage time distribution function $F_T(t|y,\dot{y})$ of the oscillator (9.54) in case of deterministic start at $(y,\dot{y}) \in S$ is considered. Instead of $F_T(t|y,\dot{y})$ calculations are performed for the *reliability function* $R(t|y,\dot{y}) = 1 - F_T(t|y,\dot{y})$.

For the present problem the initial- and boundary value problem then becomes, cf. (2.74), (6.38)

$$\left.\begin{array}{lll} \frac{\partial}{\partial t}R(t|y,\dot{y}) - \dot{y}\frac{\partial}{\partial y}R(t|y,\dot{y}) + & & \\ g(y,\dot{y})\frac{\partial}{\partial \dot{y}}R(t|y,\dot{y}) - \frac{d}{2}\frac{\partial^2}{\partial \dot{y}^2}R(t|y,\dot{y}) = 0 & , \ \forall\, t \in]0,\infty[& , \ \forall\,(y,\dot{y}) \in S \\ R(0|y,\dot{y}) = 1 & & , \ \forall\,(y,\dot{y}) \in S \\ R(t|y,\dot{y}) = 0 & , \ \forall\, t \in]0,\infty[& , \ \forall\,(y,\dot{y}) \in \partial S^{(1)} \cup \partial S^{(2)} \end{array}\right\} \tag{9.72}$$

(9.72) is an example of the initial- and boundary value problem (9.18) for the backward Kolmogorov equation, which next will be solved numerically by means of the Petrov-Galerkin variational method. (9.72) has been illustrated in fig. 9.12. As seen, the element division is the same as in fig. 9.7a. However, the principle for the node numbering is slightly changed. At the numerical solution of (9.72) the shape and weighting functions (9.59) are still used. However, since the sign of the convection terms have

changed in comparison to the diffusion terms the sign of the upwind differencing parameters α_1 and α_2 must also be changed. α_1 and the local Peclet number Pe_2 are now given by

$$\alpha_1 = \begin{cases} -1 & , \ \dot{y}_0 > 0 \\ 1 & , \ \dot{y}_0 < 0 \end{cases} \tag{9.73}$$

$$Pe_2 = \frac{g(y_0, \dot{y}_0)\Delta\dot{y}}{d} \tag{9.74}$$

α_2 then follows from (9.63). The element "mass"- and "stiffness" matrices for the backward Kolmogorov equation, $\mathbf{m}_{BK}$ and $\mathbf{k}_{BK}$, become, cf. (2.73)

$$\mathbf{m}_{\mathrm{BK}} = \left[m_{ij}\right] = \Delta y \Delta\dot{y}\left[\int_0^1\int_0^1 N_j(y,\dot{y})V_i(y,\dot{y})d\xi_1 d\xi_2\right] = \mathbf{m}_{\mathrm{FP}} \tag{9.75}$$

$$\mathbf{k}_{\mathrm{BK}} = \left[k_{ij}\right] = \Delta y \Delta\dot{y}\left[\int_0^1\int_0^1 N_j(y,\dot{y})\mathcal{K}_{\mathbf{z}}[V_i(y,\dot{y})]d\xi_1 d\xi_2\right] =$$

$$\left[\Delta y\Delta\dot{y}\int_0^1\int_0^1 N_j(y,\dot{y})\left(-\dot{y}\frac{\partial}{\partial y}V_i(y,\dot{y}) + \frac{\partial}{\partial\dot{y}}\left(g(y,\dot{y})V_i(y,\dot{y})\right) + \frac{d}{2}\frac{\partial^2}{\partial\dot{y}^2}V_i(y,\dot{y})\right)d\xi_1 d\xi_2\right] =$$

$$\left[\Delta y\Delta\dot{y}\int_0^1\int_0^1 N_j(y,\dot{y})\left(-\dot{y}\frac{\partial}{\partial y}V_i(y,\dot{y}) + \frac{\partial}{\partial\dot{y}}\left(g(y,\dot{y})V_i(y,\dot{y})\right)\right)d\xi_1 d\xi_2 -\right.$$

$$\left.\Delta y\Delta\dot{y}\int_0^1\int_0^1 \frac{d}{2}\frac{\partial}{\partial\dot{y}}N_j(y,\dot{y})\frac{\partial}{\partial\dot{y}}V_i(y,\dot{y})d\xi_1 d\xi_2\right] =$$

$$\frac{\partial}{\partial\dot{y}}g(y_0,\dot{y}_0)\frac{\Delta y\Delta\dot{y}}{36}\mathbf{m}_0 - \dot{y}_0\frac{\Delta\dot{y}}{12}\mathbf{k}_1 + g(y_0,\dot{y}_0)\frac{\Delta y}{12}\mathbf{k}_2 - d\frac{\Delta y}{12\Delta\dot{y}}\mathbf{k}_3 \tag{9.76}$$

As a consequence of the change of variable from t to $-t$ in (6.38), $-\mathcal{K}_{\mathbf{z}}^T[R(t|y,\dot{y})]$ enters on the left-hand side of (9.72) instead of $\mathcal{K}_{\mathbf{z}}^T[R(t|y,\dot{y})]$ as supposed in (9.18). For this reason the sign in front of the first integral in (9.76) has been changed in comparison to (9.21). Again, integration by parts has been performed in order to remove the second derivatives with respect to $\dot{y}$ in the last integral, and $\dot{y}$ and $g(y,\dot{y})$ have been considered constantly equal to the values in the centroid as was the case at the evaluation of (9.66). $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ are still given by (9.67), (9.68) and (9.69). As seen the local "mass" matrices for the Fokker-Planck and Kolmogorov operators are identical, whereas the local "stiffness" matrices are different.

The procedure for assembling the global "mass" and "stiffness" matrices are completely identical to the Fokker-Planck case. (9.70) should now be solved with the initial values $R(0|y_1,\dot{y}_1) = 1, \ldots, R(0|y_{N_1(N_2-1)-1}, \dot{y}_{N_1(N_2-1)-1}) = 1$.

9.2 Solution of the forward and backward Kolmogorov-Feller equations

The oscillator (9.54) is considered again, but is now excited by a compound Poisson process. Still, the double-sided the solution domain $S = \{(y,\dot{y}) \mid]a,b[\times]-\infty,\infty[\}$ is considered. The initial- and boundary value problem (9.4) then becomes, cf. (2.87)

$$\left.\begin{array}{lll} \frac{\partial}{\partial t}f(y,\dot{y},t) = -\dot{y}\frac{\partial}{\partial y}f(y,\dot{y},t) + & & \\ \frac{\partial}{\partial \dot{y}}\big(g(y,\dot{y})f(y,\dot{y},t)\big) + & & \\ \nu(t)\int\limits_{\mathcal{P}}\Big(f(y,\dot{y}-p,t)-f(y,\dot{y},t)\Big)f_P(p)dp & , \ \forall\, t\in]0,\infty[& , \ \forall\,(y,\dot{y})\in S \\ f(y,\dot{y},0) = f_0(y,\dot{y}) & & , \ \forall\,(y,\dot{y})\in S \\ f(y,\dot{y},t) = 0 & , \ \forall\, t\in]0,\infty[& , \ \forall\,(y,\dot{y})\in \partial S^{(0)}\cup\partial S^{(2)} \end{array}\right\} \tag{9.77}$$

Again, (9.77) is illustrated in fig. 9.7a. Initially, boundary conditions and the smoothness requirements of the variational field $\delta v(y,\dot{y})$ is determined in order that the property (9.5) is fulfilled. Integration by parts in the y- and $\dot{y}$-directions provides

$$\begin{aligned} &\int\limits_a^b\int\limits_{-\infty}^{\infty}\delta v(y,\dot{y})\Big(-\dot{y}\frac{\partial}{\partial y}f(y,\dot{y},t)+\frac{\partial}{\partial\dot{y}}\Big(g(y,\dot{y})f(y,\dot{y},t)\Big)\Big)dyd\dot{y} \ = \\ &\int\limits_a^b\int\limits_{-\infty}^{\infty}f(y,\dot{y},t)\Big(\dot{y}\frac{\partial}{\partial y}\delta v(y,\dot{y})-g(y,\dot{y})\frac{\partial}{\partial\dot{y}}\delta v(y,\dot{y})\Big)dyd\dot{y} \end{aligned} \tag{9.78}$$

At the integration by parts in the y-direction it is required that $\delta v(y,\dot{y}) = 0$ on $\partial S^{(1)}$ in order to cancel the boundary terms. This illustrates the remark subsequent to eq. (9.5) that $\delta v(y,\dot{y})$ is required to cancel on the part of the boundary surface, where no boundary conditions are prescribed for $f(y,\dot{y},t)$. Further, the operations leading to (9.78) require $\delta v(y,\dot{y})$ to have piecewise continuous 1st derivatives with respect to y and $\dot{y}$. Upon reversing the order of integrations and change of integration variable the following identity is obtained

$$\begin{aligned} &\int\limits_{-\infty}^{\infty}\delta v(y,\dot{y})\Big(\int\limits_{-\infty}^{\infty}f(y,\dot{y}-p,t)f_P(p)dp\Big)d\dot{y} = \int\limits_{-\infty}^{\infty}\Big(\int\limits_{-\infty}^{\infty}\delta v(y,\dot{y})f(y,\dot{y}-p,t)d\dot{y}\Big)f_P(p)dp \ = \\ &\int\limits_{-\infty}^{\infty}\Big(\int\limits_{-\infty}^{\infty}\delta v(y,\dot{x}+p)f(y,\dot{x},t)d\dot{x}\Big)f_P(p)dp = \int\limits_{-\infty}^{\infty}f(y,\dot{x},t)\Big(\int\limits_{-\infty}^{\infty}\delta v(y,\dot{x}+p)f_P(p)dp\Big)d\dot{x} \end{aligned} \tag{9.79}$$

Notice, p is constant at the integral substition $\dot{x} = \dot{y}-p$ in the innermost integral of the second statement of (9.79). The operations involved in (9.79) merely require $\delta v(y,\dot{y})$ to

be piecewise constant and $|\delta v(y,\dot{y})|$ to be integrable over the interval $]-\infty,\infty[$. Upon combining the results (9.78) and (9.79) the identity (9.7) follows, cf. (2.86). Then, the indicated smoothness requirements to the variational field is seen to be fulfilled if the shape- and weighting functions are taken on the following product form

$$N_{ij}(y,\dot{y}) = N(\xi_i)N(\eta_j) \ , \ V_{ij}(y,\dot{y}) = \Big(N(\xi_i)+\alpha_1 W(\xi_i)\Big)\Big(N(\eta_j)+\alpha_2 W(\eta_j)\Big) \tag{9.80}$$

where

$$\xi_i = \frac{y-y_i}{\Delta y} \ , \ \eta_j = \frac{\dot{y}-\dot{y}_j}{\Delta \dot{y}} \tag{9.81}$$

Now the previously defined double indexing of system nodes is resumed. $N(\cdot)$ and $W(\cdot)$ are the continuous piecewise linear and quadratic shape- and upwind differencing functions shown in figs. 9.1a and 9.3a, respectively. Because the shape of the functions are independent of the node the indices on these have been omitted for ease. The specification of the upwind differencing parameters α_1 and α_2 are postponed until later. (9.10), (9.11) and (9.12) are rewritten in the following form, cf. (2.86)

$$\sum_{i_2=1}^{N_1+1}\sum_{j_2=1}^{N_2+1} M_{i_1 j_1 i_2 j_2}\dot{f}_{i_2 j_2}(t) - \sum_{i_2=1}^{N_1+1}\sum_{j_2=1}^{N_2+1} K_{i_1 j_1 i_2 j_2} f_{i_2 j_2}(t) = 0 \tag{9.82}$$

$$M_{i_1 j_1 i_2 j_2} = \int_a^b \int_{-\infty}^{\infty} N_{i_2 j_2}(y,\dot{y}) V_{i_1 j_1}(y,\dot{y}) dy d\dot{y} \tag{9.83}$$

$$K_{i_1 j_1 i_2 j_2} = \int_a^b \int_{-\infty}^{\infty} N_{i_2 j_2}(y,\dot{y}) \mathcal{K}^T_{\mathbf{z},t}[V_{i_1 j_1}(y,\dot{y})] dy d\dot{y} =$$

$$\int_a^b \int_{-\infty}^{\infty} N_{i_2 j_2}(y,\dot{y}) \Big(\dot{y}\frac{\partial}{\partial y} V_{i_1 j_1}(y,\dot{y}) - g(y,\dot{y})\frac{\partial}{\partial \dot{y}} V_{i_1 j_1}(y,\dot{y}) +$$

$$\nu(t) \int_{\mathcal{P}} \Big(V_{i_1 j_1}(y,\dot{y}+p) - V_{i_1 j_1}(y,\dot{y}) \Big) f_P(p) dp \Big) dy d\dot{y} = K^{(1)}_{i_1 j_1 i_2 j_2} + K^{(2)}_{i_1 j_1 i_2 j_2} \tag{9.84}$$

$$K^{(1)}_{i_1 j_1 i_2 j_2} = \int_a^b \int_{-\infty}^{\infty} N_{i_2 j_2}(y,\dot{y}) \Big(\dot{y}\frac{\partial}{\partial y} V_{i_1 j_1}(y,\dot{y}) - g(y,\dot{y})\frac{\partial}{\partial \dot{y}} V_{i_1 j_1}(y,\dot{y}) - \nu(t) V_{i_1 j_1}(y,\dot{y}) \Big) dy d\dot{y} \tag{9.85}$$

$$K^{(2)}_{i_1 j_1 i_2 j_2} = \nu(t) \int_a^b \int_{-\infty}^{\infty} N_{i_2 j_2}(y,\dot{y}) \Big(\int_{\mathcal{P}} V_{i_1 j_1}(y,\dot{y}+p) f_P(p) dp \Big) dy d\dot{y} \ =$$

$$\nu(t)\int_a^b N(\xi_{i_2})\Big(N(\xi_{i_1})+\alpha_1 W(\xi_{i_1})\Big)dy\cdot\int_{-\infty}^{\infty} N(\eta_{j_2})\Big(n^T(\eta_{j_1})+\alpha_2 w^T(\eta_{j_1})\Big)d\dot{y} \;=$$

$$\frac{\nu(t)}{6}\Delta y\Delta\dot{y}\cdot\gamma_{i_1-i_2}\cdot\delta^T_{j_1-j_2} \tag{9.86}$$

$$n^T(\eta_j)=\int_{\mathcal{P}} N\Big(\frac{\dot{y}-\dot{y}_j+p}{\Delta\dot{y}}\Big)f_P(p)dp \tag{9.87}$$

$$w^T(\eta_j)=\int_{\mathcal{P}} W\Big(\frac{\dot{y}-\dot{y}_j+p}{\Delta\dot{y}}\Big)f_P(p)dp \tag{9.88}$$

$$\gamma_k=\int_{-1}^{1} N(\xi)\Big(N(\xi-k)+\alpha_1 W(\xi-k)\Big)d\xi=\begin{cases}1+\frac{3}{2}\alpha_1 & , \quad k=-1\\ 4 & , \quad k=0\\ 1-\frac{3}{2}\alpha_1 & , \quad k=1\\ 0 & , \quad \text{elsewhere}\end{cases} \tag{9.89}$$

$$\delta^T_k=\int_{-1}^{1} N(\eta)\Big(n^T(\eta-k)+\alpha_2 w^T(\eta-k)\Big)d\eta \quad , \quad k=0,\pm1,\pm2,\ldots \tag{9.90}$$

where it has been used that $\xi_{i_1}=\xi_{i_2}-(i_1-i_2)$, $\eta_{j_1}=\eta_{j_2}-(j_1-j_2)$, cf. (9.81). The evaluation of the integral in (9.89) follows from (9.44), (9.45). The 4th order tensors $M_{i_1j_1i_2j_2}$ and $K_{i_1j_1i_2j_2}$ merely specify with an extended indexing the components of the previous 2nd order tensors M_{ij} and K_{ij}. M_{ij} and $K^{(1)}_{ij}$ can be assembled by the finite element method, where $K^{(1)}_{ij}$ signifies the 2nd order "stiffness" tensor corresponding to the 4th order tensor $K^{(1)}_{i_1j_1i_2j_2}$. The element "mass" matrix of the forward Kolmogorov-Feller operator $\mathbf{m}_{\mathrm{FKF}}=\mathbf{m}_{\mathrm{FP}}$ is unchanged given by (9.64). The element stiffness matrix corresponding to $K^{(1)}_{ij}$ is given by

$$\mathbf{k}^{(1)}_{\mathrm{FKF}}=\dot{y}_0\frac{\Delta\dot{y}}{12}\mathbf{k}_1-g(y_0,\dot{y}_0)\frac{\Delta y}{12}\mathbf{k}_2-\nu(t)\frac{\Delta y\Delta\dot{y}}{36}\mathbf{m}_0 \tag{9.91}$$

where $\mathbf{m}_0$, $\mathbf{k}_1$, $\mathbf{k}_2$ are given by (9.65), (9.67), (9.68).

The functions $n^T(\eta_j)$ and $w^T(\eta_j)$ have not support in a compact set centered around $\dot{y}_j$, as is the case for the corresponding non-convoluted functions $N(\eta_j)$ and $W(\eta_j)$. In principle these functions then implies coupling in the $\dot{y}$-direction among all layers of element (unless the sample space of P is bounded) as follows fron the equivalent finite difference stencil shown below in fig. 9.14. For this reason a finite element representation of $K^{(2)}_{i_1j_1i_2j_2}$ cannot be given. Instead, the coefficients δ^T_k given by (9.90) must be

evaluated numerically for each combination of the indices j_1 and j_2. As an example let $P \sim N(0, \sigma_P^2)$. Then the following results can be derived for $n^T(\eta)$ and $w^T(\eta)$

$$n^T(\eta,\beta) = \beta\left(\Psi_1\left(\frac{\eta-1}{\beta}\right) - 2\Psi_1\left(\frac{\eta}{\beta}\right) + \Psi_1\left(\frac{\eta+1}{\beta}\right)\right) \tag{9.92}$$

$$w^T(\eta,\beta) = -3\beta^2\left(\Psi_2\left(\frac{\eta-1}{\beta}\right) - 2\Psi_2\left(\frac{\eta}{\beta}\right) + \Psi_2\left(\frac{\eta+1}{\beta}\right)\right) - 3\beta\left(\Psi_1\left(\frac{\eta-1}{\beta}\right) - \Psi_1\left(\frac{\eta+1}{\beta}\right)\right) \tag{9.93}$$

where

$$\beta = \frac{\sigma_P}{\Delta\dot{y}} \tag{9.94}$$

$$\Psi_1(x) = x\Phi(x) + \varphi(x) \quad , \quad \Psi_2(x) = (1+x^2)\Phi(x) + x\varphi(x) \tag{9.95}$$

$\Phi(\cdot)$ and $\varphi(\cdot)$ signify the probability distribution function and the probability density function of a standardized normal variable. Below in fig. 9.13 the analytical solutions (9.92) and (9.93) have been shown for various value of the nondimensional standard deviation β of the impulse strength.

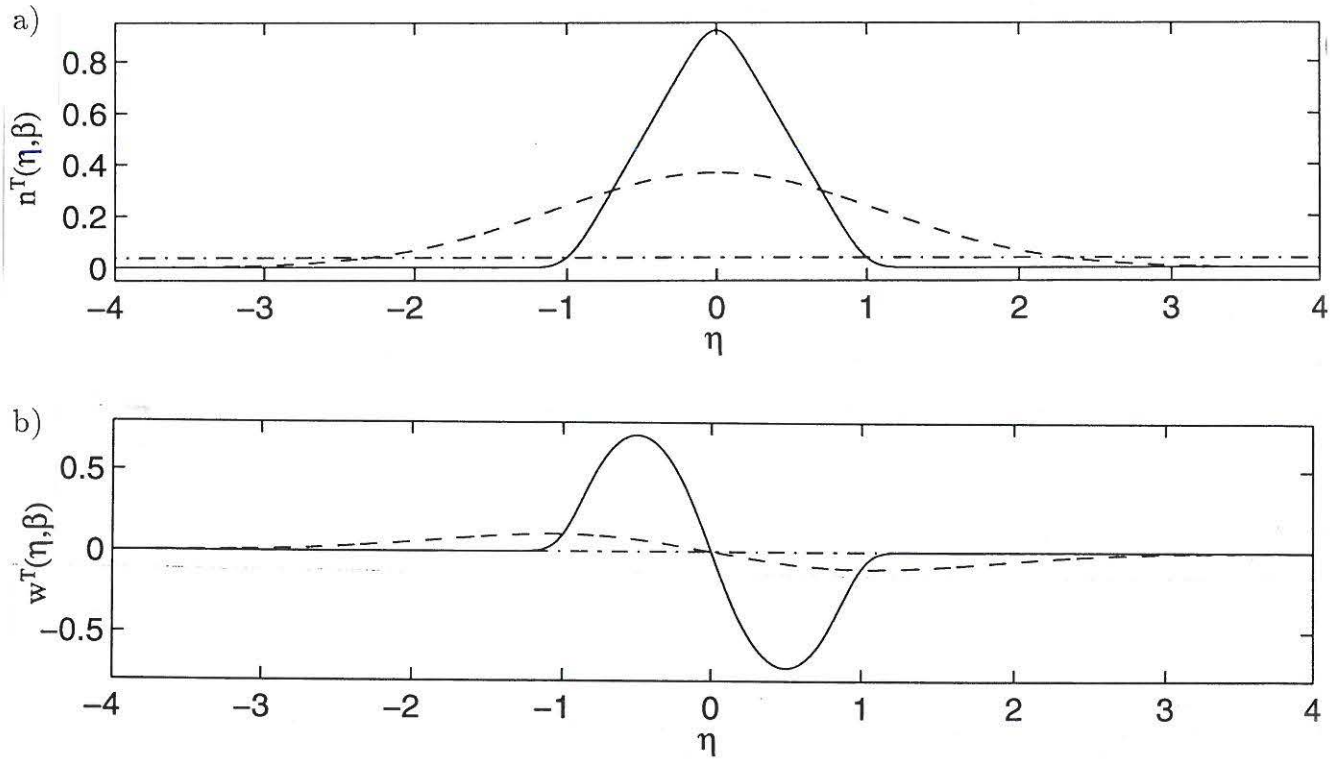

Fig. 9.13: $n^T(\eta,\beta)$ and $w^T(\eta,\beta)$ as functions of β and η. a) $n^T(\eta,\beta)$. b) $w^T(\eta,\beta)$. $\beta = 0.1$: —- , $\beta = 1$: - - - - , $\beta = 10$: -.-.- .

For $\beta = 0.1$, $n^T(\eta, \beta)$ and $w^T(\eta, \beta)$ have strong ressemblance to the corresponding non-convoluted functions. Hence the coupling to other than the neighbouring element layers is insignificant in this case. However, for $\beta = 1$ and especially for $\beta = 10$ coupling among larger parts of the mesh takes place. Besides, as β is increased the upwinding effect in the $\dot{y}$-direction is seen to disappear. Since long range couplings should be omitted for numerical reasons, the lesson learned from this example is, that β should be selected rather small, say less than 1.0. For a given magnitude of the impulses as measured by the standard deviation σ_P, this imposes a lower limit on the element width $\Delta\dot{y}$ as follows from (9.94). Especially, for the case of sparse pulse trains where σ_P is relatively large, this effect should be taken into consideration.

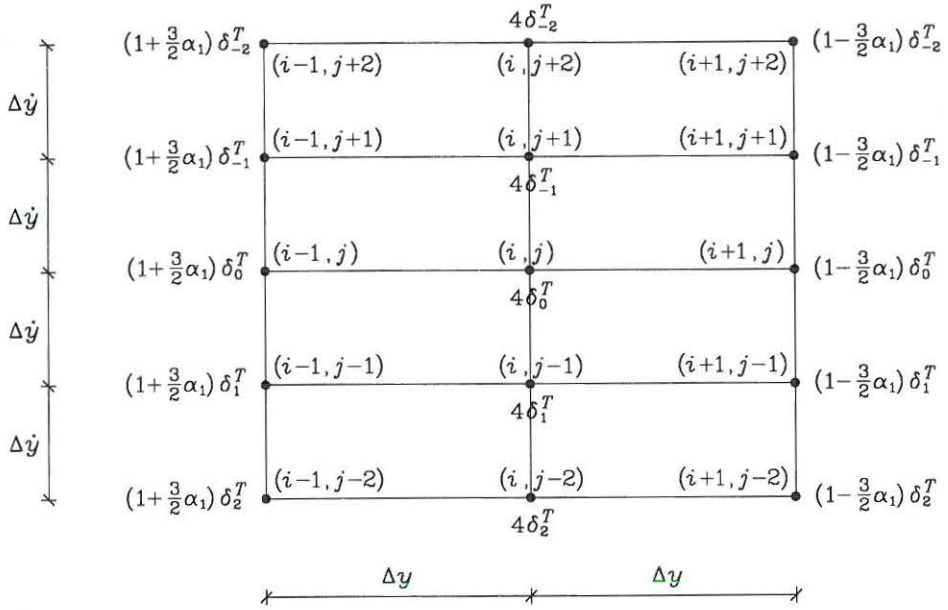

Fig. 9.14: Equivalent finite difference scheme for $\sum_{i_2=1}^{N_1+1}\sum_{j_2=1}^{N_2+1} K^{(2)}_{i_1 j_1 i_2 j_2} f_{i_2 j_2}(t)$. Common factor: $\frac{\nu}{6}\Delta y \Delta\dot{y}$.

In order to calculate a local Peclet number for convection and diffusion in the $\dot{y}$-direction a Taylor expansion of $f(y, \dot{y}-p, t)$ in p is performed. The right-hand side of (9.77) then becomes

$$-\dot{y}\frac{\partial}{\partial y}f(y,\dot{y},t)+\frac{\partial}{\partial\dot{y}}\big(g(y,\dot{y})f(y,\dot{y},t)\big)+\nu(t)\int_{\mathcal{P}}\Big(f(y,\dot{y}-p,t)-f(y,\dot{y},t)\Big)f_P(p)dp\;=$$

$$-\dot{y}\frac{\partial}{\partial y}f(y,\dot{y},t)+\frac{\partial}{\partial\dot{y}}\big(g(y,\dot{y})f(y,\dot{y},t)\big)-\nu(t)E[P]\frac{\partial}{\partial\dot{y}}f(y,\dot{y},t)+$$

$$\frac{1}{2!}\nu(t)E[P^2]\frac{\partial^2}{\partial\dot{y}^2}f(y,\dot{y},t)+\nu(t)\sum_{n=3}^{\infty}\frac{(-1)^n}{n!}E[P^n]\frac{\partial^n}{\partial\dot{y}^n}f(y,\dot{y},t) \qquad (9.96)$$

(9.96) is the Kramer-Moyal expansion of the forward Kolmogorov-Feller operator, cf. (2.87). Obviously, this expansion requires that moments of arbitrary order $E[P^n]$ exists,

which is not necessary the case for the left-hand side of (9.96). From (9.96) follows that an equivalent convection velocity in the $\dot{y}$-direction can be defined as $-g(y,\dot{y})+\nu(t)E[P]$ and an equivalent diffusion coefficient as $\nu(t)E[P^2]$. The local Peclet number in the $\dot{y}$-direction is then calculated from, cf. (9.61)

$$Pe_2 = -\frac{\Big(g(y_0,\dot{y}_0) - \nu(t)E[P]\Big)\Delta\dot{y}}{\nu(t)E[P^2]} \tag{9.97}$$

Next, the upwind parameter α_2 is calculated from (9.63). The local Peclet number and upwind parameter α_1 in the y-direction are unchanged given by (9.60) and (9.62).

Next, the boundary and initial value problem (6.38) for the determination of the reliability function is considered, cf. (2.7x), (6.3x)

$$\left.\begin{array}{l}
\frac{\partial}{\partial t}R(t|y,\dot{y}) - \dot{y}\frac{\partial}{\partial y}R(t|y,\dot{y}) + \\
g(y,\dot{y})\frac{\partial}{\partial \dot{y}}R(t|y,\dot{y}) - \\
\nu(t)\int\limits_{\mathcal{P}}\Big(R(t|y,\dot{y}+p) - R(t|y,\dot{y})\Big)f_P(p)dp = 0 \quad , \quad \forall\, t \in]0,\infty[\quad , \quad \forall\,(y,\dot{y}) \in S \\
R(0|y,\dot{y}) = 1 \qquad\qquad , \quad \forall\,(y,\dot{y}) \in S \\
R(t|y,\dot{y}) = 0 \qquad\qquad , \quad \forall\, t \in]0,\infty[\quad , \quad \forall\,(y,\dot{y}) \in \partial S^{(1)} \cup \partial S^{(2)}
\end{array}\right\} \tag{9.98}$$

Again, (9.98) is illustrated in fig. 9.12, and the shape- and weighting function are given by (9.80). The 4th order "mass" tensor is unchanged given by (9.83), whereas the 4th order "stiffness" tensor becomes, cf. the remarks subsequent to (9.76)

$$\begin{aligned}
K_{i_1j_1i_2j_2} &= \int\limits_a^b\int\limits_{-\infty}^{\infty} N_{i_2j_2}(y,\dot{y})\mathcal{K}_{\mathbf{z}}[V_{i_1j_1}(y,\dot{y})]dyd\dot{y} = \\
&\int\limits_a^b\int\limits_{-\infty}^{\infty} N_{i_2j_2}(y,\dot{y})\Big(-\dot{y}\frac{\partial}{\partial y}V_{i_1j_1}(y,\dot{y}) + \frac{\partial}{\partial \dot{y}}\Big(g(y,\dot{y})V_{i_1j_1}(y,\dot{y})\Big) + \\
&\nu(t)\int\limits_{\mathcal{P}}\Big(V_{i_1j_1}(y,\dot{y}-p) - V_{i_1j_1}(y,\dot{y})\Big)f_P(p)dp\Big)dyd\dot{y} = K^{(1)}_{i_1j_1i_2j_2} + K^{(2)}_{i_1j_1i_2j_2}
\end{aligned} \tag{9.99}$$

$$K^{(1)}_{i_1j_1i_2j_2} = \int\limits_a^b\int\limits_{-\infty}^{\infty} N_{i_2j_2}(y,\dot{y})\Big(-\dot{y}\frac{\partial}{\partial y}V_{i_1j_1}(y,\dot{y}) + \frac{\partial}{\partial \dot{y}}\Big(g(y,\dot{y})V_{i_1j_1}(y,\dot{y})\Big) - \nu(t)V_{i_1j_1}(y,\dot{y})\Big)dyd\dot{y} \tag{9.100}$$

$$K^{(2)}_{i_1j_1i_2j_2} = \nu(t)\int\limits_a^b\int\limits_{-\infty}^{\infty} N_{i_2j_2}(y,\dot{y})\Big(\int_{\mathcal{P}} V_{i_1j_1}(y,\dot{y}-p)f_P(p)dp\Big)dyd\dot{y} \;=$$

$$\nu(t)\int_a^b N(\xi_{i_2})\Big(N(\xi_{i_1})+\alpha_1 W(\xi_{i_1})\Big)dy\cdot\int_{-\infty}^{\infty} N(\eta_{j_2})\Big(n(\eta_{j_1})+\alpha_2 w(\eta_{j_1})\Big)d\dot{y} \tag{9.101}$$

$$n(\eta_j)=\int_{\mathcal{P}} N\Big(\frac{\dot{y}-\dot{y}_j-p}{\Delta\dot{y}}\Big)f_P(p)dp \tag{9.102}$$

$$w(\eta_j)=\int_{\mathcal{P}} W\Big(\frac{\dot{y}-\dot{y}_j-p}{\Delta\dot{y}}\Big)f_P(p)dp \tag{9.103}$$

Generally, the functions $n(\eta)$ and $w(\eta)$ are different from the functions $n^T(\eta)$ and $w^T(\eta)$ given by (9.87) and (9.88), unless the symmetry property $f_P(p)=f_P(-p)$ is fulfilled. Especially, if $P\sim N(0,\sigma_P^2)$ then $n(\eta)$ and $w(\eta)$ are given by (9.92) and (9.93). As was the case for the forward Kolmogorov-Feller operator the 4th order tensors $M_{i_1j_1i_2j_2}$ and $K^{(1)}_{i_1j_1i_2j_2}$ can be assembled by the finite element method to the equivalent 2nd order tensors M_{ij} and K_{ij}. The element "mass" matrix of the backward Kolmogorov-Feller operator $\mathbf{m}_{\mathrm{BKF}}=\mathbf{m}_{\mathrm{FP}}$ is unchanged given by (9.64). The element stiffness matrix corresponding to $K^{(1)}_{ij}$ is given by

$$\mathbf{k}^{(1)}_{\mathrm{BKF}}=-\dot{y}_0\frac{\Delta\dot{y}}{12}\mathbf{k}_1+g(y_0,\dot{y}_0)\frac{\Delta y}{12}\mathbf{k}_2-\Big(\nu(t)-\frac{\partial}{\partial\dot{y}}g(y_0,\dot{y}_0)\Big)\frac{\Delta y\Delta\dot{y}}{36}\mathbf{m}_0 \tag{9.104}$$

where $\mathbf{m}_0$, $\mathbf{k}_1$, $\mathbf{k}_2$ are given by (9.65), (9.67), (9.68). Still, the tensor components $K^{(2)}_{i_1j_1i_2j_2}$ are assembled into the global "stifness" matrix according to the grid mesh depicted in fig. 9.11. However, δ_k^T should be replaced by the coefficients

$$\delta_k=\int_{-1}^{1} N(\eta)\Big(n(\eta-k)+\alpha_2 w(\eta-k)\Big)d\eta\quad,\quad k=0,\pm1,\pm2,\ldots \tag{9.105}$$

Upwind differencing parameters should change in sign in comparison to the specifications for the forward Kolmogorov-Keller equation, similar to the change of signs of the upwind differencing parameters of the the backward Kolmogorov equation in comparison to those of the Fokker-Planck equation, cf. (9.73) and (9.74). α_1 is unchanged given by (9.73), whereas the local Peclet number Pe_2 is given as

$$Pe_2=\frac{\Big(g(y_0,\dot{y}_0)-\nu(t)E[P]\Big)\Delta\dot{y}}{\nu(t)E[P^2]} \tag{9.106}$$

9.3 Concluding remarks

In this section a Petrov-Galerkin variational formulation of the forward and backward Kolmogorov equations have been presented for one- and two-dimensional diffusion and convection problems. The aim has been to use a formulation based on the finite element method for the solution of these problems. This is not possible in case of pulse driven systems, where a mixed formulation is presented. Whenever it works the Petrov-Galerkin variational methods provide results of the same level of accuracy as do the comparable cell-to-cell mapping method. In contrast it has not been possible to devise a formulation of the Petrov-Galerkin method useable for pulse problems with sparse pulse arrivals, as was possible for the cell-to-cell mapping scheme in section 8.2. Both methods shares the drawback (as do the moment equation methods for that matter) that they become intractable very fast as the dimension of the state vector is increased. Numerical solutions for state vectors of dimension larger than say 5 is out of the question with todays technology, even with parallellization of the calculations.

The first application of the finite element formulation of the Petrov-Galerkin variational method to the solution of the Fokker-Planck and the backward equation was due to Bergman and his co-workers, who in a number of papers demonstrated the applicability of the method to a number of non-hysteretic simple oscillators, Bergman and Heinrich [9.3], Bergman and Spencer [9.4], as well as hysteretic oscillators, Spencer [9.5]. Similar variational approaches have later been proposed by Langley [9.6], Langtangen [9.7], and others.

Köylüoğlu et al. [9.8] formulated a Petrov-Galerkin variational approach for the solution of the two-dimensional backward Kolmogorov-Feller equation (9.96), using the product form (9.80) of the shape and weighting functions. In order to circumvent the problem of the numerical integration in the second integral of the last statement of (9.99), a Kramer-Moyal expansion of the backward Kolmogorov-Feller operator was applied. Correspondingly, the shape and upwind differencing functions in the $\dot{y}$-direction were taken as normally distributed to meet the requirement of infinitely often differentiability. The same problems of stability of the numerical scheme were observed at sparse impulse arrival as reported in example 9.4.

9.4 Bibliography

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9.5 Example problems

9.1 Derive or verify the analytical solution (9.30).

9.2 Derive $\mathbf{K}_0(Pe,\alpha)$ and $\mathbf{M}_0(\alpha)$ for the following upwind differencing functions, $W_i(z) = -\frac{\pi}{4}\sin(\pi\xi)$, where $\xi = \frac{z-z_i}{\Delta z}$, and $W_i(z) = \frac{\Delta z}{2}\frac{d}{dz}N_i(z)$.

9.3 Derive the solution (9.50) from (9.48) and (9.49).

9.4 Show that the finite element method with the local "mass" matrix (9.64) and local "stiffness" matrix (9.66) is equivalent to a finite difference scheme with the local truncation error $\mathrm{O}(\max(\Delta y, \Delta\dot{y}))$.

9.5 Prove that the Crank-Nicholson scheme (9.70) has a local truncation error $\mathrm{O}(\Delta t^2)$, and that the scheme is unconditionally stable.

9.6 Calculate the stationary joint probability density function and the corresponding marginal probability density functions of the displacement Y and the velocity $\dot{Y}$ using the result (9.71) of a van der Pol oscillator with $g(Y,\dot{Y}) = 2\zeta(1-Y^2)\dot{Y} - Y$, $\zeta > 0$, exposed to a Gaussian white noise with $d = 4\zeta$. Use the parameter values $\zeta = 1$, $N_1 = N_2 = 30$, $-a = b = 3$, $\Delta\dot{y} = 0.3$, $\Delta t = 0.05$.

9.7 Derive the analytical solutions (9.92), (9.93).

9.8 Calculate the stationary joint probability density function and the corresponding marginal probability density functions of the displacement Y and the velocity $\dot{Y}$ using the result (9.71) of a van der Pol oscillator with $g(Y,\dot{Y}) = 2\zeta(1-Y^2)\dot{Y} - Y$, $\zeta > 0$, exposed to a stationary compound Poisson process with $\nu E[P^2] = 4\zeta$, where $P \sim N(0,\sigma_P^2)$. Use the parameter values $\nu = 0.1$, $\zeta = 1$, $N_1 = N_2 = 30$, $-a = b = 3$, $\Delta\dot{y} = 0.3$, $\Delta t = 0.05$.

CHAPTER 10

TECHNIQUES OF EQUIVALENT LINEARIZATION AND EQUIVALENT POLYNOMIAL EXPANSIONS FOR GEOMETRICALLY NON-LINEAR AND HYSTERETIC SDOF AND MDOF SYSTEMS

10.1 Introductory remarks – idea of resorting to equivalent systems

A method, or a technique, which found a wide variety of applications to different non-linear problems is the equivalent (or statistical) linearization technique. Its idea consists in considering, instead of the original system governed by a non-linear differential equation, an equivalent system governed by a linear differential equation. Such an approach was originally used to deterministic problems governed by non-linear differential equations. Next, this technique was adapted by Booton [10.1] and Kazakov [10.2] to stochastic problems of control theory, and by Caughey [10.3] to non-linear stochastic problems of structural dynamics. An extensive overview of various applications of the equivalent linearization technique is given e.g. in the references [10.4-10.6]. An essential feature of this technique is that the coefficients of the equivalent linear equation are evaluated from the condition of minimization of the mean-square difference of equations and are expressed in terms of the moments and of the expectations of non-linear functions of the response process (state variables). If the non-linearities are of polynomial (power) type then, of course, the equivalent coefficients are expressed in terms of moments only. If, however, the non-linearities are non-algebraic or even non-analytical functions of the state variables (e. g. in the case of hysteretic systems), then these expectations can only be evaluated as integrals with respect to a suitable probability measure. An important question which arises is with respect to what measure (probability density) these expectations, including moments, should be evaluated, because the exact probability density function of the response of the original system is unknown. It can be proved that if the exact density were assumed, the first and second order response moments evaluated from the equivalent linear system would be exact (of course, if the exact density were known, the linearization would be unnecessary). In practice, one would like the moments obtained from the equivalent linear system to be the most accurate possible, and therefore the response probability density function should be assumed in tentative form as close to the exact one as possible. Very often, however, this probability density function can be assumed, at best, as the probability density of the response of a linearized system. This is especially easy if the Gaussian excitation is considered, since the response of a linearized system is a Gaussian distributed process as well, and hence the tentative density function is assumed as the Gaussian one. The equations governing the first- and second-order moments derived from the equivalent linear system become non-linear, and can only be solved numerically; the equivalent coefficients are updated in each step of numerical integration. Unfortunately, if the tentative Gaussian density is assumed, the departure of the response process from Gaussianity, which is an important property of behaviour of the response of a non-linear system cannot be investigated. Despite this inherent shortcoming, the equivalent linearization technique

yields in many cases the reliable estimates of the response mean value and variance, and therefore it is often applied, especially to multi-degree-of-freedom systems if the first- and second-order moments are required [10.5, 10.6].

As a generalization of the idea of the equivalent linearization the technique of an equivalent non-linear equation (or system) was proposed by Caughey (1986) [10.7], in which instead of an original non-linear system, another non-linear system is considered, for which the exact solution is known. Equivalent coefficients of a non-linear equation are evaluated from the mean-square optimization condition. This technique allows to investigate the departure of response probability distribution from the Gaussian one. Its range of applicability is, however, rather restricted, mainly due to the fact that the class of non-linear systems for which the exact solution exists, is narrow.

A particular version of the technique of the equivalent non-linear equation is the technique of equivalent polynomial expansion (or polynomial form). In this technique an equivalent non-linear system, is considered, in which the original, often non-analytical, non-linear functions of the state variables are substituted by polynomials in these variables, for example by a quadratic form (equivalent quadratization) or by a cubic form (equivalent cubicization) [10.8, 10.9]. Coefficients of the equivalent polynomial form are evaluated from the mean-square optimization condition, i.e. from the condition of minimization, in the mean-square sense, of the equation difference and are expressed in terms of expectations of non-linear functions of the state variables. Of course in order to perform the necessary expectations the unknown response probability density must be assumed in a tentative form, usually as a Gram-Charlier expansion. It can be proved that if an exact response probability density were assumed, an equivalent non-linear system with nth order polynomial, would yield the exact moments up to and including $n + 1$ order [10.8, 10.9]. Equivalent equations are solved with the help of known methods, for example with the help of the moment equations technique combined with non-Gaussian closure approximation [10.8, 10.9]. Equations for moments of the response of the equivalent non-linear system are obviously non-linear and must be solved numerically. This technique allows in many cases to obtain much more exact results than the equivalent linearization technique.

The earliest application of the equivalent linearization technique to the problem of random impulses is due to Tylikowski and Marowski [10.10] who considered a Duffing oscillator under a Poisson impulsive noise. Since in the considered problem the non-linearities were of cubic form, the required expectations were just the higher order moments, which were substituted by the moments of a linearized system. In that case, if the linearization is performed in the steady-state, the moments are constant and the equivalent system is time-invariant, hence its response moments can be evaluated exactly from the known formulae as the moments of a filtered Poisson process, without assuming any tentative form of the joint probability density function. However the range of applicability of such an approach is confined to systems with polynomial non-linearities only. Recently Grigoriu [10.11] used a similar approach for non-linear systems under Poisson impulses.

The equivalent linearization and cubicization techniques have been further developed for

hysteretic systems under Poisson impulses [10.12, 10.13]. Since in this case the response probability density is not known, not even in the case of a linear system, the tentative, non-Gaussian approximate probability density of the state variables is be assumed in form of a Gram-Charlier expansion. The equations for moments up to fourth order have been derived for equivalent systems; these moments were used to determine the expansion coefficients. Comparison of analytically obtained (computed) response mean values and variances with simulated ones revealed good accuracy of the approximate analytical techniques developed, especially of the technique of equivalent cubicization [10.12, 10.13].

10.2 Equivalent linearization and polynomial expansion technique for Gaussian white noise driven systems

10.2.1 Equivalent linearization technique

SDOF systems

To illustrate the idea, let us consider a non-linear oscillator, governed by the equation

$$m\big(\ddot{Y} + \mathcal{G}(Y,\dot{Y})\big) = F(t) \tag{10.1}$$

where $\mathcal{G}(Y,\dot{Y}) = g(Y,\dot{Y}) + \frac{\partial}{\partial Y}U(Y)$, cf. (1.86) and $F(t)$ is a Gaussian white noise process.

Instead of the original non-linear system governed by the equation (10.1) consider a linear system, governed by the equation

$$m\big(\ddot{Y} + \beta_e\dot{Y} + k_eY\big) = F(t) \tag{10.2}$$

where β_e, k_e are the unknown coefficients to be determined.

The exact solution to the equation (10.1) is unknown. The only obtainable solution is the one to the linear equation (10.2). Upon insertion of this solution into the equation (10.1) the left- and right-hand sides of (10.1) differ by some $\epsilon(Y,\dot{Y})$, which is the so-called equation difference, i.e.

$$\epsilon(Y,\dot{Y}) = \mathcal{G}(Y,\dot{Y}) - \beta_e\dot{Y} - k_eY \tag{10.3}$$

It is required that the linear system be equivalent in some sense to the original one. The criterion usually assumed is that the mean square of the equation difference be minimized with respect to the coefficients β_e, k_e termed as *the equivalent coefficients.* Hence the necessary minimization condition are

$$\frac{\partial}{\partial\beta_e}E\left[\epsilon^2\right] = 0, \qquad \frac{\partial}{\partial k_e}E\left[\epsilon^2\right] = 0 \tag{10.4}$$

After performing the necessary operations the expressions for the equivalent coefficients are arrived at in the form of

$$\beta_e = \frac{E[Y^2]E[\dot{Y}\mathcal{G}(Y,\dot{Y})] - E[Y\dot{Y}]E[Y\mathcal{G}(Y,\dot{Y})]}{E[Y^2]E[\dot{Y}^2] - (E[Y\dot{Y}])^2} \tag{10.5}$$

$$k_e = \frac{E[\dot{Y}^2]E[Y\mathcal{G}(Y,\dot{Y})] - E[Y\dot{Y}]E[\dot{Y}\mathcal{G}(Y,\dot{Y})]}{E[Y^2]E[\dot{Y}^2] - (E[Y\dot{Y}])^2} \tag{10.6}$$

These are implicit expressions for β_e, k_e, since any expectations at the right-hand sides of (10.5) and (10.6) can only be evaluated based on the linear equation (10.2) and hence they also depend on β_e and k_e.

If the transient, non-stationary response is considered, all the expectations entering the right-hand sides of equations (10.5) and (10.6) are time-dependent, and consequently the equivalent coefficients are functions of time. If however, the linearization is effectuated in the steady state and the excitation is stationary, then also the response process is stationary, hence all these expectations are constant in time and so are the equivalent coefficients. Moreover, since the displacement response $Y(t)$ and the velocity response $\dot{Y}(t)$ are then uncorrelated, i.e. $E[Y\dot{Y}] = 0$, the expressions (10.5) and (10.6) simplify to

$$\beta_e = \frac{E[\dot{Y}\mathcal{G}(Y,\dot{Y})]}{E[\dot{Y}^2]} \tag{10.7}$$

$$k_e = \frac{E[Y\mathcal{G}(Y,\dot{Y})]}{E[Y^2]} \tag{10.8}$$

If the non-linear function $\mathcal{G}(Y,\dot{Y})$ is of polynomial form, as it is in the case of systems with geometrical non-linearities and e.g. in the case of Duffing (1.87), Rayleigh (1.88) or van der Pol (1.89) oscillators, the expectations entering the right-hand sides of equations (10.5)-(10.8) are expressed directly in terms of the moments. These moments can be evaluated from the linearized system.

In the case of systems with, non-analytical non-linerities, e.g. hysteretic systems, the expectations entering the right-hand sides of equations (10.5)-(10.8) can only be evaluated by performing the integrals with respect to the tentative probability density function.

If, further, $Y(t)$ and $\dot{Y}(t)$ are assumed to be jointly Gaussian distributed, the following relationship which holds for a Gaussian random variable $X(t)$, can be used

$$E[Xf(X)] = E[X^2]E\left[\frac{\partial f(X)}{\partial X}\right] \tag{10.9}$$

where $f(X)$ is a non-linear function.

Then the equivalent coefficients are given by

$$\beta_e = E\left[\frac{\partial \mathcal{G}(Y,\dot{Y})}{\partial \dot{Y}}\right] \tag{10.10}$$

$$k_e = E\left[\frac{\partial \mathcal{G}(Y,\dot{Y})}{\partial Y}\right] \tag{10.11}$$

In principle, the problem lies in assuming a tentative joint probability density function of the response $[Y(t), \dot{Y}(t)]$, or in other words, the probability measure with respect to which the expectations in (10.4) and all subsequent expectations are evaluated. Assuming a Gaussian density function as a tentative density function is, in the case of Gaussian excitation, tantamount to assuming the probability density of the response of a linearized system (10.2). It is obvious that if such an assumption is made, no insight can be gained into the non-Gaussian behaviour of the response of the original non-linear system (10.1).

MDOF systems

Consider the non-linear MDOF system with purely external single excitation, i.e. the diffusion vector $\mathbf{d}(t)$ is assumed to be independent of the state vector $\mathbf{Z}(t)$. Moreover consider the general case of a non-zero-mean excitation $\mathbf{F}(t)$ represented as the sum of its mean value $E[\mathbf{F}(t)] = \boldsymbol{\mu}_{\mathbf{F}}(t)$ and the zero-mean Gaussian white noise. Thus the governing stochastic equations are written as

$$d\mathbf{Z}(t) = \mathbf{c}(\mathbf{Z}(t), t)dt + \boldsymbol{\mu}_{\mathbf{F}}(t)dt + \mathbf{d}(t)dW(t) \tag{10.12}$$

The equations for mean values are, cf. (3.39)

$$\frac{d}{dt}\boldsymbol{\mu}(t) = E\left[\mathbf{c}(\mathbf{Z}(t), t)\right] + \boldsymbol{\mu}_{\mathbf{F}}(t) \tag{10.13}$$

The equations for the centralized state variables become, cf. (3.40)

$$d\mathbf{Z}^0(t) = \mathbf{c}^0(\mathbf{Z}^0(t), t)dt + \mathbf{d}(t)dW(t) \tag{10.14}$$

where

$$\mathbf{c}^0(\mathbf{Z}^0(t), t) = \mathbf{c}(\mathbf{Z}^0(t), t) - E[\mathbf{c}(\mathbf{Z}^0(t), t)] \tag{10.15}$$

Instead of an original system governed by (10.14), consider now a linear system, governed by equations

$$d\mathbf{Z}^0(t) = \mathbf{B}\mathbf{Z}^0(t)dt + \mathbf{d}(t)dW(t) \tag{10.16}$$

where the centralized drift term is assumed as a linear form in the state variables

$$c^0_{i,eq}(\mathbf{Z}^0(t)) = B_{im} Z^0_m \tag{10.17}$$

and B_{im} are the coefficients to be determined.

The solution $\mathbf{Z}^0(t)$ of equation (10.16) satisfies the original equation with some error

$$\varepsilon\left(\mathbf{Z}^0\right) = \mathbf{c}^0\left(\mathbf{Z}^0\right) - \mathbf{B}Z^0 \tag{10.18}$$

It is that this error be minimized in the mean-square sense. The necessary conditions of the minimum are

$$\frac{\partial}{\partial B_{im}} E\left[\varepsilon^T\left(\mathbf{Z}^0\right)\varepsilon\left(\mathbf{Z}^0\right)\right] = 0 \tag{10.19}$$

It follows that the equivalent coefficients B_{im} of the linear form satisfy the algebraic equations

$$B_{im}\kappa_{mj} = E\left[Z^0_j c^0_i\left(\mathbf{Z}^0\right)\right] \tag{10.20}$$

which can be represented in the matrix form of

$$\mathbf{B}\boldsymbol{\kappa}(t) = E\left[\mathbf{c}^0(\mathbf{Z}^0(t))\mathbf{Z}^{0^T}\right] \tag{10.21}$$

If, further, the state variables $\mathbf{Z}^0(t)$ are assumed to be jointly Gaussian distributed, the following relationship which holds for a zero-mean Gaussian random vector $\mathbf{X}$, can be used (Atalik and Utku [10.16])

$$E[\mathbf{X}f(\mathbf{X})] = E[\mathbf{X}\mathbf{X}^T]E\left[\nabla f(\mathbf{X})\right] \tag{10.22}$$

where $\nabla = \left[\frac{\partial}{\partial X_1}, \frac{\partial}{\partial X_2}, \ldots, \frac{\partial}{\partial X_n}\right]^T$ and $f(\mathbf{X})$ is a non-linear function.

Transposing both sides of equation (10.21) yields

$$\boldsymbol{\kappa}^T(t)\mathbf{B}^T = \boldsymbol{\kappa}(t)\mathbf{B}^T = E\left[\mathbf{Z}^0\mathbf{c}^{0^T}(\mathbf{Z}(t))\right] \tag{10.23}$$

and using the result (10.22) gives

$$\boldsymbol{\kappa}(t)\mathbf{B}^T = \boldsymbol{\kappa}(t)E\left[\nabla\mathbf{c}^{0^T}(\mathbf{Z}^0(t))\right] \tag{10.24}$$

hence the equivalent coefficients are evaluated as

$$\mathbf{B}^T = E\left[\nabla\mathbf{c}^{0^T}(\mathbf{Z}^0(t))\right] \tag{10.25}$$

or more explicitly [10.15], [10.16]

$$B_{ij} = E\left[\frac{\partial c_i^0(\mathbf{Z}^0)}{\partial Z_j^0}\right] \tag{10.26}$$

The second-order moments are then evaluated from the equations

$$\frac{d}{dt}\kappa_{ij}(t) = 2\{B_{im}\kappa_{mj}\}_s + d_i d_j \tag{10.27}$$

or, in the matrix form

$$\frac{d}{dt}\boldsymbol{\kappa}(t) = \mathbf{B}\boldsymbol{\kappa} + \boldsymbol{\kappa}\mathbf{B}^T + \mathbf{d}\mathbf{d}^T \tag{10.28}$$

Two important facts should be mentioned in connection with the equivalent linearization technique. The first one is that since the equivalent coefficients are evaluated based on the condition of minimization of the mean square of the equation difference and Gaussian distribution assumption is used to perform the expectations (10.20), (10.21), the second order moments obtained from the equivalent linear system are exactly the same as the ones obtained from the original equations for moments truncated with the help of Gaussian closure.

Notice that the expectations of the type (10.20) are present in the equations for the second order moments of the original system, see (3.44). If the Gaussian closure is applied to the equations (3.44), the Gaussian vector relationship (10.22) is used, hence the expectation is evaluated as

$$E\left[Z_j^0 c_i^0\left(\mathbf{Z}^0\right)\right] = E\left[\frac{\partial c_i^0(\mathbf{Z}^0)}{\partial Z_m^0}\right]\kappa_{mj} \tag{10.29}$$

The equation for the second order moments becomes

$$\frac{d}{dt}\kappa_{ij}(t) = 2\left\{E\left[\frac{\partial c_i^0(\mathbf{Z}^0)}{\partial Z_m^0}\right]\kappa_{mj}\right\}_s + d_i d_j \tag{10.30}$$

Exactly the same equation is obtained in the case of the linearized system if in (10.27) the following substitution is made

$$B_{im} = E\left[\frac{\partial c_i^0(\mathbf{Z}^0)}{\partial Z_m^0}\right] \tag{10.31}$$

The second important observation is that if the exact joint probability density function was used to perform the expectations in (10.20) the exact second order moments would be obtained from the equivalent system. It is obvious that performing the expectations

in the equations (3.44) for the original system with respect to the exact joint probability density function provides exact second order moments. Comparing equations (10.27) and (10.20) shows that if the exact joint probability density function was used, the equation (10.27) would take the same form as (3.44), hence it would yield the exact second order moments.

10.2.2 Equivalent polynomial expansion technique

SDOF systems

To illustrate the idea of the equivalent polynomial expansion, consider the non-linear oscillator governed by the equation

$$m\left(\ddot{Y} + 2\zeta\omega_0\dot{Y} + \mathcal{G}(Y)\right) = F(t) \tag{10.32}$$

Instead of the original non-linear system governed by equation (10.32) consider another non-linear system, which will be called equivalent, in which the non-linearity is given by a polynomial, e.g. by the third order polynomial

$$m\left(\ddot{Y} + 2\zeta\omega_0\dot{Y} + k_1Y + k_2Y^2 + k_3Y^3\right) = F(t) \tag{10.33}$$

where k_1, k_2, k_3 are the unknown coefficients to be determined.

Proceeding as in the case of equivalent linearization we obtain the equation difference as

$$\epsilon(Y) = \mathcal{G}(Y) - k_1Y - k_2Y^2 - k_3Y^3 \tag{10.34}$$

It is required that the system governed by (10.33) be equivalent to the original non-linear system in the mean-square sense. The necessary minimization conditions are

$$\frac{\partial}{\partial k_1}E\left[\epsilon^2\right] = 0, \qquad \frac{\partial}{\partial k_2}E\left[\epsilon^2\right] = 0, \qquad \frac{\partial}{\partial k_3}E\left[\epsilon^2\right] = 0 \tag{10.35}$$

These conditions lead to the following algebraic equations for k_1, k_2, k_3

$$\begin{aligned}
k_1E[Y^2] &= E[Y\mathcal{G}(Y)] - k_2E[Y^3] - k_3E[Y^4] \\
k_2E[Y^4] &= E[Y^2\mathcal{G}(Y)] - k_1E[Y^3] - k_3E[Y^5] \\
k_3E[Y^6] &= E[Y^3\mathcal{G}(Y)] - k_1E[Y^4] - k_2E[Y^5]
\end{aligned} \tag{10.36}$$

which can be rearranged as

$$\begin{bmatrix} E[Y^2] & E[Y^3] & E[Y^4] \\ E[Y^3] & E[Y^4] & E[Y^5] \\ E[Y^4] & E[Y^5] & E[Y^6] \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} E[Y\mathcal{G}(Y)] \\ E[Y^2\mathcal{G}(Y)] \\ E[Y^3\mathcal{G}(Y)] \end{bmatrix} \tag{10.37}$$

MDOF systems

Equations for the mean responses remain the same, i.e. (10.13), and the equations for the centralized state variables are of the form (10.14).

Here, instead of the original non-linear system (10.14) an equivalent non-linear system is considered in which the non-linear functions are assumed in polynomial form. For the sake of simplicity let us confine the attention to a cubic form. Hence this system is governed by the equations

$$dZ_i^0(t) = A_i + B_{im}Z_m^0 + C_{imn}Z_m^0 Z_n^0 + D_{imnp}Z_m^0 Z_n^0 Z_p^0 + d_i(t)dW(t) \tag{10.38}$$

The centralized drift term has been assumed as the cubic form in the state variables

$$c_{i,eq}^0(\mathbf{Z}^0(t),t) = A_i + B_{im}Z_m^0 + C_{imn}Z_m^0 Z_n^0 + D_{imnp}Z_m^0 Z_n^0 Z_p^0 \tag{10.39}$$

The unknown equivalent coefficients are evaluated based on the minimization of the mean square of the difference between the equations (10.14) for the original system and the equations (10.38) for the equivalent one, i.e.

$$\boldsymbol{\varepsilon}\left(\mathbf{Z}^0\right) = \mathbf{c}^0\left(\mathbf{Z}^0\right) - \mathbf{c}_{eq}^0(\mathbf{Z}^0(t),t) \tag{10.40}$$

The necessary minimum conditions are

$$\begin{aligned} &\frac{\partial}{\partial A_i}E\left[\boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon}\right] = 0 \quad , \quad \frac{\partial}{\partial B_{im}}E\left[\boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon}\right] = 0 \\ &\frac{\partial}{\partial C_{imn}}E\left[\boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon}\right] = 0 \quad , \quad \frac{\partial}{\partial D_{imnp}}E\left[\boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon}\right] = 0 \end{aligned} \tag{10.41}$$

These conditions lead to the following algebraic equations

$$\begin{aligned} &A_i + C_{imn}\kappa_{mn} + D_{imnp}\kappa_{mnp} = 0 \\ &B_{im}\kappa_{mj} + C_{imn}\kappa_{mnj} + D_{imnp}\kappa_{mnpj} = E\left[Z_j^0 c_i^0(\mathbf{Z}^0)\right] \\ &A_i\kappa_{jk} + B_{im}\kappa_{mjk} + C_{imn}\kappa_{mnjk} + D_{imnp}\kappa_{mnpjk} = E\left[Z_j^0 Z_k^0 c_i^0(\mathbf{Z}^0)\right] \\ &A_i\kappa_{jkl} + B_{im}\kappa_{mjkl} + C_{imn}\kappa_{mnjkl} + D_{imnp}\kappa_{mnpjkl} = E\left[Z_j^0 Z_k^0 Z_l^0 c_i^0(\mathbf{Z}^0)\right] \end{aligned} \tag{10.42}$$

The expectations at the right-hand side of equations (10.42) are the same as those entering the equations for moments, up to the fourth order, of the original system (3.44). This implies two important conclusions.

Firstly, if the exact joint probability density function was used to evaluate the expectations at the right-hand side of equations (3.44) the exact moments would be obtained.

Comparison of equations (3.44) and (3.46) with (10.42) reveals that if the exact joint probability density function was also used to evaluate the expectations at the right-hand side of (10.42), the equations for the moments, and hence the moments themselves, up to the fourth order for the equivalent cubic system are then the same as the ones for the original system. In general, if the rth order polynomial expansion was used and the exact joint probability density function was used to evaluate the pertinent expectations, moments up to the order $r+1$ would be evaluated exactly from the equivalent system, cf. [10.9], [10.17].

Secondly, if the same tentative joint probability density function was used to evaluate the expectations at the right-hand side of equations (10.42) and in the equations for moments (3.44), the same moments up to the fourth order would be obtained. Generalizing this observation one can state that in the case of the rth order polynomial expansion the same moments up to the $r+1$ order would be obtained if the same tentative joint probability density function was used to evaluate the expectations in the equivalent polynomial expansion technique and to perform the expectation in the equation for moments of the original system, cf. [10.9], [10.17].

The joint probability density function of the state variables $Z_j^0(t)$ which is needed to perform the expectations in the equations (10.42) will be assumed approximately in the form of a truncated Gram-Charlier expansion constructed from the moments of the order up to the fourth, cf. (3.63)

$$f_{\{\mathbf{Z}\}}(\mathbf{z},t) = \phi(\mathbf{z}) \left\{ 1 + \frac{1}{3!} \sum_{i,j,k} \beta_{ijk}(t) H_{ijk}(\mathbf{z}) + \frac{1}{4!} \sum_{i,j,k,l} \beta_{ijkl}(t) H_{ijkl}(\mathbf{z}) \right\} \tag{10.43}$$

The moments of orders up to the fourth have to be obtained from the equations for moments of the equivalent non-linear system (with cubic non-linearity in the considered case) (3.46).

It should be noted that all the expectations in the equations (10.42), as they result from (10.41), are performed with respect to the same density function, hence the fifth- and sixth-order moments in (10.42), must also be evaluated by integration with respect to the assumed density function (10.43). This yields the expression for the fifth-order moments the same as in (3.64), and for the sixth-order moments (3.66) (quasi-moment closure). Obviously the known moments, i.e. those of orders up to the fourth, which appear in the equations for moments (3.46) and in (10.42) are the same.

If the cumulant neglect closure were used to evaluate the redundant fifth- and sixth-order moments which appear in (10.42), then the expectations which appear at the right-hand sides of (10.42) would have to be evaluated as integrals with respect to the density function which corresponds to the cumulant neglect closure. In this case the truncated Gram-Charlier expansion constructed from the moments of the order up to the fourth becomes

$$f_{\{\mathbf{Z}\}}(\mathbf{z},t) = \phi(\mathbf{z}) \left\{ 1 + \frac{1}{3!} \sum_{i,j,k} \beta_{ijk}(t) H_{ijk}(\mathbf{z}) + \frac{1}{4!} \sum_{i,j,k,l} \beta_{ijkl}(t) H_{ijkl}(\mathbf{z}) + \right.$$

$$\frac{1}{6!}\sum_{i,j,k,l,m,n} 10\left\{\lambda_{ijk}(t)\lambda_{lmn}(t)\right\}_s H_{ijklmn}(\mathbf{z})\Bigg\} \tag{10.44}$$

All the redundant fifth- and sixth-order moments which appear in the equations for moments (3.46) may be estimated with the help of arbitrary closure approximations.

10.3 Equivalent linearization and polynomial expansion technique for Poisson impulse process driven systems

10.3.1 Modified equivalent linearization technique

Consider again the non-linear MDOF system with purely external excitation, i.e. the vector $\mathbf{b}$ (see equation (5.5) is constant and is assumed to be independent of the state vector $\mathbf{Z}(t)$.

Governing stochastic equations take the form of

$$d\mathbf{Z}(t) = \mathbf{c}\left(\mathbf{Z}(t),t\right)dt + \mathbf{b}\int_{\mathcal{P}} pM(dt,t,dp,p) \tag{10.45}$$

Equations for the mean values $E\left[\mathbf{Z}(t)\right] = \boldsymbol{\mu}(t)$ in the form of

$$\frac{d}{dt}\boldsymbol{\mu}(t) = E\big[\mathbf{c}(\mathbf{Z}(t),t)\big] + \nu(t)E[P]\mathbf{b} \tag{10.46}$$

Equations for the centralized state variables (zero-mean response processes) $Z_i^0(t) = Z_i(t) - \mu_i(t)$ are obtained as

$$d\mathbf{Z}^0(t) = \mathbf{c}^0\left(\mathbf{Z}^0(t),t\right)dt + \mathbf{b}\int_{\mathcal{P}} pM(dt,t,dp,p) \tag{10.47}$$

where

$$\mathbf{c}^0\left(\mathbf{Z}^0(t),t\right) = \mathbf{c}\left(\mathbf{Z}^0(t)+\boldsymbol{\mu}(t),t\right) - E\left[\mathbf{c}\left(\mathbf{Z}^0(t)+\boldsymbol{\mu}(t),t\right)\right] - \nu(t)E[P]\mathbf{b} \tag{10.48}$$

Instead of the original system, governed by the equations (10.47), let us consider a linear system governed by the equations

$$d\mathbf{Z}^0(t) = \left\{\mathbf{B}Z^0(t) - \nu(t)E[P]\mathbf{b}\right\}dt + \mathbf{b}\int_{\mathcal{P}} pM(dt,t,dp,p) \tag{10.49}$$

obtained by substituting the centralized drift term

$$c_i^0\left(\mathbf{Z}^0+\boldsymbol{\mu}\right)=c_i\left(\mathbf{Z}^0+\boldsymbol{\mu}\right)-E\left[c_i\left(\mathbf{Z}^0+\boldsymbol{\mu}\right)\right]$$

by an equivalent linear form

$$c_{i,eq}^0\left(\mathbf{Z}^0+\boldsymbol{\mu}\right)=B_{im}Z_m^0 \tag{10.50}$$

The mean square minimization of the equation error leads to the algebraic equations for the equivalent coefficients B_{im} of the linear form

$$B_{im}\kappa_{mj}=E\left[Z_j^0c_i^0\left(\mathbf{Z}^0+\boldsymbol{\mu}\right)\right] \tag{10.51}$$

where

$$E\left[Z_j^0c_i^0\left(\mathbf{Z}^0+\boldsymbol{\mu}\right)\right]=E\left[Z_j^0c_i\left(\mathbf{Z}^0+\boldsymbol{\mu}\right)\right] \tag{10.52}$$

Here, however, a Gaussian distribution assumption about the vector $\mathbf{Z}^0(t)$ cannot be made. Moreover, in the present problem, unlike the Gaussian excitation problem, the probability density of the response of a linear system cannot be used as a tentative density function in order to perform the expectations appearing at the right-hand side of eq. (10.51), because this density is unknown. Obviously, the tentative probability density function should be assumed in a non-Gaussian form. To do that it is expedient to use the truncated Gram-Charlier expansion (10.43), for which the moments up to the fourth order are required. Equations for moments of the response of the linearized system, derived based on the stochastic equation (10.49), become, cf. (5.45)-(5.47)

$$\dot{\kappa}_{ij}(t)=2\left\{B_{im}\kappa_{mj}\right\}_s+\nu(t)E\left[P^2\right]b_ib_j \tag{10.53}$$

$$\dot{\kappa}_{ijk}(t)=3\left\{B_{im}\kappa_{mjk}\right\}_s+\nu(t)E\left[P^3\right]b_ib_jb_k \tag{10.54}$$

$$\dot{\kappa}_{ijkl}(t)=4\left\{B_{im}\kappa_{mjkl}\right\}_s+\nu(t)E\left[P^2\right]6\left\{b_ib_j\kappa_{kl}\right\}_s+\nu(t)E\left[P^4\right]b_ib_jb_kb_l \tag{10.55}$$

10.3.2 Equivalent cubic form (equivalent cubicization) technique

Equations for the mean responses remain the same, i.e. (10.46). Here, instead of the original non-linear system (10.47) an equivalent non-linear system is considered, governed by the equation

$$dZ_i^0(t)=\left\{A_i+B_{im}Z_m^0+C_{imn}Z_m^0Z_n^0+D_{imnp}Z_m^0Z_n^0Z_p^0-\nu(t)E[P]b_i\right\}dt+$$

$$b_i\int_{\mathcal{P}}pM(dt,t,dp,p) \tag{10.56}$$

In this case the original centralized drift term $c_i^0(\mathbf{Z}^0 + \boldsymbol{\mu})$ has been substituted by a cubic form in the state variables

$$c_{i,eq}^0(\mathbf{Z}^0 + \boldsymbol{\mu}) = A_i + B_{im}Z_m^0 + C_{imn}Z_m^0 Z_n^0 + D_{imnp}Z_m^0 Z_n^0 Z_p^0 \tag{10.57}$$

Equations for the moments of the response of an equivalent system with cubic non-linearity have the form of (5.45)-(5.47).

The difference between the equation (10.47) for an original system and the equation (10.56) for an equivalent system is

$$\boldsymbol{\varepsilon}(\mathbf{Z}^0 + \boldsymbol{\mu}) = \mathbf{c}^0(\mathbf{Z}^0 + \boldsymbol{\mu}) - \mathbf{c}_{eq}^0(\mathbf{Z}^0 + \boldsymbol{\mu}) \tag{10.58}$$

The equivalent coefficients of a cubic form (10.57) are evaluated based on the condition that the equation difference (10.58) be minimized, i.e.

$$\begin{aligned} &\frac{\partial}{\partial A_i} E\left[\boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon}\right] = 0, \quad \frac{\partial}{\partial B_{im}} E\left[\boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon}\right] = 0 \\ &\frac{\partial}{\partial C_{imn}} E\left[\boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon}\right] = 0, \quad \frac{\partial}{\partial D_{imnp}} E\left[\boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon}\right] = 0 \end{aligned} \tag{10.59}$$

These conditions lead to the following algebraic equations

$$\begin{aligned} A_i &= -C_{imn}\kappa_{mn} - D_{imnp}\kappa_{mnp} \\ B_{im}\kappa_{mj} &= E\left[Z_j^0 c_i^0(\mathbf{Z}^0)\right] - C_{imn}\kappa_{mnj} - D_{imnp}\kappa_{mnpj} \\ C_{imn}\kappa_{mnjk} &= E\left[Z_j^0 Z_k^0 c_i^0(\mathbf{Z}^0)\right] - A_i\kappa_{jk} - B_{im}\kappa_{mjk} - D_{imnp}\kappa_{mnpjk} \\ D_{imnp}\kappa_{mnpjkl} &= E\left[Z_j^0 Z_k^0 Z_l^0 c_i^0(\mathbf{Z}^0)\right] - A_i\kappa_{jkl} - B_{im}\kappa_{mjkl} - C_{imn}\kappa_{mnjkl} \end{aligned} \tag{10.60}$$

The joint probability density function of the state variables which is needed to perform the expectations in the equations (10.60) could be assumed tentatively as a probability density of the response of an equivalent non-linear system. Since this density function is unknown, it will be assumed approximately in the form of a truncated Gram-Charlier expansion (10.43) constructed from the moments of the order up to the 4th. It should be noted that all the expectations in the equations (10.60) are performed with respect to the same density function as in (10.59), hence the 5th and 6th order moments in (10.60), must also be evaluated by integration with respect to the density function (10.43). This yields the expression for the 5th order moments (3.65), and for the 6th order moments (3.66). Obviously the moments of the order up to the 4th, which appear in the equations (5.45)-(5.47) and (10.60) are the same. In order to construct the expansion (10.43) the moments of the order up to the fourth are required. All the redundant 5th and 6th order moments which appear in the equations for moments (5.45)-(5.47) should be evaluated with the help of some closure approximations. To do that any closure technique can be used, for example the cumulant-neglect closure (3.65).

Example 10.1: Single-degree-of-freedom hysteretic system

Consider a single-degree-of-freedom hysteretic system under a Poisson train of general pulses with the displacement Z_1 and velocity Z_2. Let the hysteretic component Z_3 of the restoring force be governed by equation of the form (1.96) and let the general pulses be idealized as in the section 5.3.3 with Z_4 and Z_5 being the filter displacement and velocity, respectively.

Moreover let us introduce a damage variable. As is well known, the plastic deformations are associated by the material deterioration or damage development. As a variable accounting for the damage, the hysteretic energy $Z_6(t) = D(t)$ can be assumed [10.18], which is just the hysteresis loop area. The time evolution of this variable is governed by the equation [10.18]

$$\dot{D}(t) = \dot{Z}_1 Z_3 \tag{10.61}$$

Then the drift and diffusion vectors become, respectively

$$\mathbf{c}(\mathbf{Z}) = \begin{bmatrix} Z_2 \\ -\alpha\Omega_0^2 Z_1 - 2\zeta\Omega_0 Z_2 - (1-\alpha)\Omega_0^2 Z_3 + Z_4 \\ aZ_2 - \beta \mid Z_2 \mid Z_3 \mid Z_3 \mid^{n-1} - \gamma Z_2 \mid Z_3 \mid^n \\ Z_5 \\ -\Omega_f^2 Z_4 - 2\zeta_f \Omega_f Z_5 \\ Z_2 Z_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{10.62}$$

The data assumed for the hysteretic system is: $n = 1$, $a = 1$, $\beta = \gamma = 0.5$, $\alpha = 0.05$, $\zeta = 0.01$ $\Omega_0 = 1[\mathrm{s}^{-1}]$ and for the auxiliary linear filter: $\Omega_0 = 1[\mathrm{s}^{-1}]$, $\zeta_f = 0.95$. The mean arrival rate of the pulses is $\nu = 1[\mathrm{s}^{-1}]$ and the impulses of the driving train are assumed as non-zero-mean Rayleigh distributed random variables characterized by $\sigma_P = 3.70834$.

Analytical results for the mean values and variances of the state variables obtained from the modified equivalent linearization and equivalent cubic expansion techniques have been verified against Monte-Carlo simulations based on the averaging over the ensemble of 50, 000 independent response sample functions.

The modified equivalent linearization technique yields good estimates of the mean values (Figs. 10.1 - 10.3), but the estimates of the variances are not satisfactory. Only the transient maximum of $Var(Z_1)$ is well predicted (Fig. 10.4). The prediction of $Var(Z_3)$ (Fig. 10.5) is poor (maybe only qualitatively correct) and the prediction of $Var(Z_6)$ (Fig. 10.6) is completely wrong.

In contrast, the equivalent cubic expansion (equivalent cubicization) technique provides much better accuracy of the results: also the predictions of the variances are accurate enough.

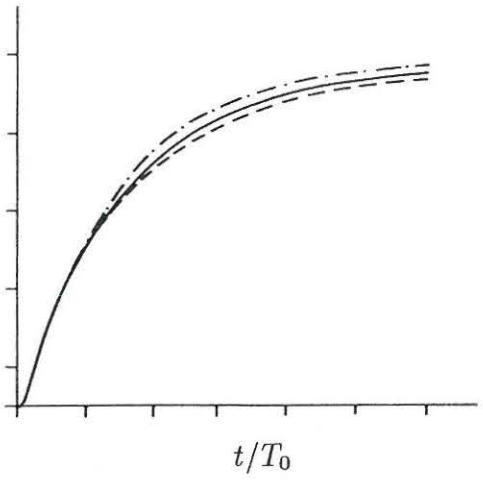

Fig. 10.1. Mean displacement response of a hysteretic system to a Poissonian train of general pulses: – · – · – · modified equivalent linearization – – – equivalent cubic form, —— simulated.

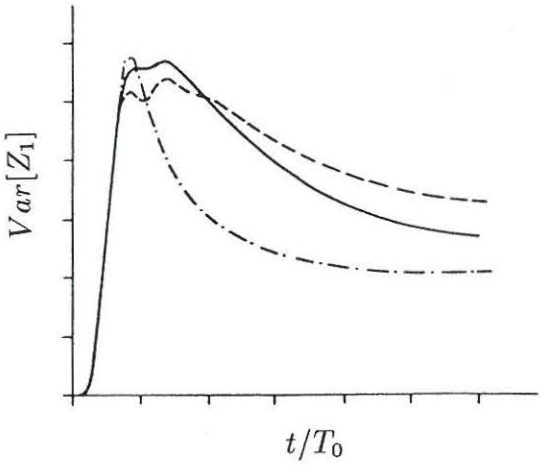

Fig. 10.4. Displacement response variance in a hysteretic system subjected to a Poissonian train of general pulses: see Fig. 10.1.

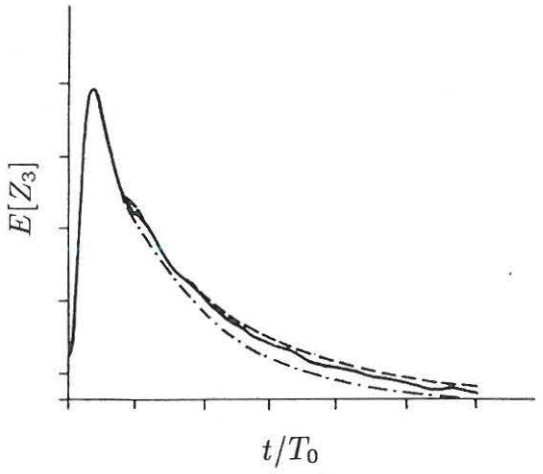

Fig. 10.2. Mean hysteretic restoring force in a system subjected to a Poissonian train of general pulses: see Fig. 10.1.

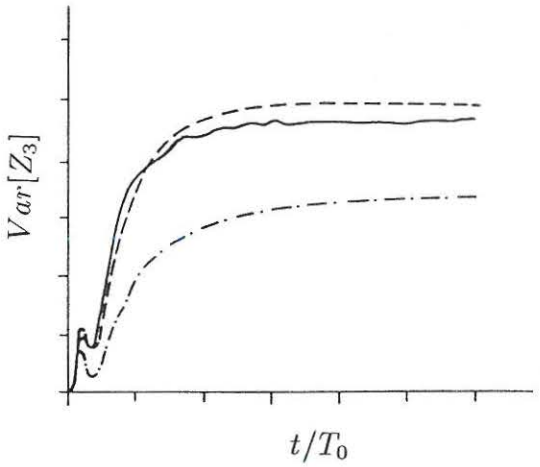

Fig. 10.5. Hysteretic restoring force variance in a system subjected to a Poissonian train of general pulses: see Fig. 10.1.

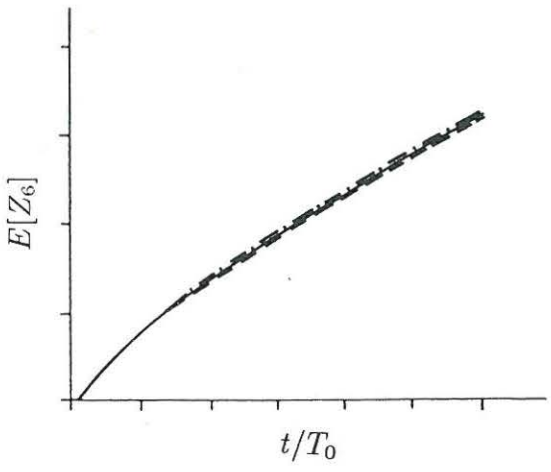

Fig. 10.3. Mean damage indicator in a hysteretic system subjected to a Poissonian train of general pulses: see Fig. 10.1.

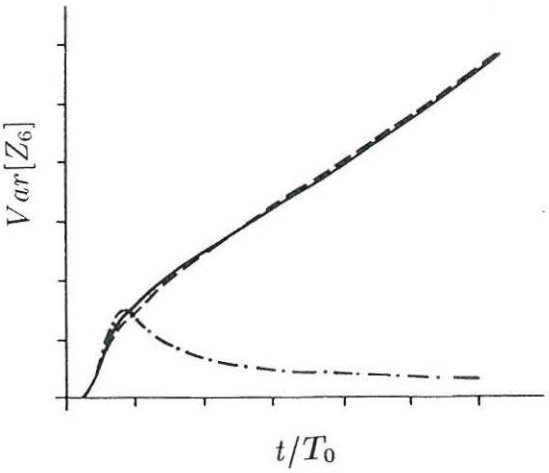

Fig. 10.6. Variance of the damage indicator in a hysteretic system subjected to a Poissonian train of Dirac delta zero-mean impulses: see Fig. 10.1.

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12. SUBJECT INDEX